Fixed Points and Stability of an Integral Equation: Nonuniqueness

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ABSTRACT. We consider a paper of Banaś and Rzepka which deals with existence and asymptotic stability of an integral equation by means of fixed point theory and measures of noncompactness. By choosing a different fixed point theorem we show that the measures of noncompactness can be avoided and the existence and stability can be proved under weaker conditions. Moreover, we show that this is actually a problem about a bound on the behavior of a nonunique solution. In fact, without nonuniqueness, the theorems of stability are vacuous.

Key words: Fixed points, stability, integral equations, nonuniqueness.

1. Introduction. Recently, Banaś and Rzepka [1] studied an integral equation by means of a modification of a fixed point theorem of Darbo using measures of noncompactness. They obtained two interesting results and two examples on existence and stability.

The purpose of this paper is twofold. First, investigators have found that a more careful selection of a fixed point theorem and mapping set can eliminate the need for studies of noncompactness; we illustrate that in Theorem 3 and Theorem 6. Along the same line we eliminate their condition for boundedness and we isolate the function to which solutions converge. Next, the authors do not mention nonuniqueness and both of their examples possess unique solutions; yet it will be pointed out that their stability results are nonvacuous if and only if their equation has a non-unique solution. In the case of nonuniqueness, those results turn out to be very important for they show that, while a solution may be nonunique, those solutions branching off will return and approach the solution from which they diverged. In effect, then, the nonuniqueness may not be catastrophic, as it is in the case of a differential equation such as $x'(t) = x^{1/3}(t)$ where solutions break off from the zero solution and go off to infinity. Their work is for a scalar equation, but ours is for a vector system.

2. Results of stability and boundedness. In [1] Banaś and Rzepka consider an operator F mapping the space of bounded continuous functions $BC(R_+)$ into itself, $R_+ = [0, \infty)$, such that

$$(1) \qquad |(Fx)(t) - (Fy)(t)| \le k|x(t) - y(t)| + a(t)$$

for all $x, y \in BC$, $t \in R_+$, where $k \in [0,1)$ and $a : R_+ \to R_+$ is continuous and $\lim_{t\to\infty} a(t) = 0$. They assume that there is an $x \in BC$ satisfying

$$x = Fx. \tag{2}$$

They then prove the following result which incorporates a non-standard definition.

THEOREM-DEFINITION 1. Under the above conditions, the function x is an asymptotically stable solution of (2): that is, for any $\epsilon > 0$ there exists $T > 0$ such that for every $t \geq T$ and for every other solution y of (2), then $|x(t) - y(t)| \leq \epsilon$.

The standard definition would ask that solutions starting arbitrarily close to the given solution remain close and converge to it. But that definition rules out nonuniqueness which is the very heart of this investigation.

They then consider the integral equation

$$x(t) = f(t, x(t)) + \int_0^t u(t, s, x(s))ds, \qquad t \geq 0, \tag{3}$$

where they assume that:

(i) $f : R_+ \times R \to R$ is continuous and $f(t, 0) \in BC(R_+)$;

(ii) there is a constant $k \in [0, 1)$ with

$$|f(t, x) - f(t, y)| \leq k|x - y| \tag{4}$$

for all $t \geq 0$ and all $x, y \in R$;

(iii) $u : R_+ \times R_+ \times R \to R$ is continuous and there are continuous functions $a, b : R_+ \to R_+$ such that

$$\lim_{t\to\infty} a(t) \int_0^t b(s)ds = 0 \tag{5}$$

and

$$|u(t, s, x)| \leq a(t)b(s) \tag{6}$$

for all $t, s \in R_+, (s \leq t)$ and all $x \in R$.

REMARK 1. Condition (ii) implies that there is one and only one point x_0 with $x_0 = f(0, x_0)$; every continuous solution $x(t)$ of (3) satisfies $x(0) = x_0$. In the stability definition there are "other solutions" only in the case of nonuniqueness. See Theorem 3 for details.

They then use fixed point theory to prove

THEOREM 2. If (i)-(iii) hold then (3) has at least one solution $x \in BC(R_+)$ and it is asymptotically stable.

First they show that a bounded continuous solution exists using fixed point theory. Then they invoke Theorem-Definition 1 to conclude that it is asymptotically stable. This is followed by the remark that the asymptotic stability will also follow from the properties of a certain set in the fixed point argument. Finally, they offer two examples;

$$x(t) = \frac{t}{1+t^2}x(t) + \int_0^t e^{-t}\frac{sx(s)}{1+|x(s)|}ds, \tag{7}$$

$$x(t) = \frac{\ln(1+t)}{1+t}\sin x(t) + \int_0^t \frac{s^2 \arctan x(s)}{1+t^4}ds. \tag{8}$$

REMARK 2. Equation (7) has the zero solution; moreover, it is Lipschitz in x so that is the only solution. Hence, neither Theorem 1 nor Theorem 2 shed any light on (7). If we take the principal branch of the arctan function in (8), then it has the unique zero solution. If we take another branch, then Theorem 2 is useful and yields a bounded solution; but it is unique and so stability holds by default.

REMARK 3. Hypothesis (i) is critical only if we require a bounded solution. We later prove that if (3) has a solution and if (ii) and (iii) hold with k replaced by $k(t)$, then that solution is bounded whenever $|f(t,0)|/(1-k(t))$ is bounded. We can prove the existence of a solution without assuming boundedness of $f(t,0)$ and the stability works in the same way.

3. A parallel theorem. Very early in the study of asymptotic stability by fixed point theory it was recognized that compactness on an infinite interval presented problems. A simple solution was to construct a mapping set which degenerated to a curve (usually the zero solution) as $t \to \infty$. This meant that an equicontinuous subset was, in fact, contained in a compact set. In a standard integral equation

$$x(t) = c(t) + \int_0^t g(t,s,x(s))ds,$$

the usual expectation is that the solution will follow $c(t)$ in some sense. In a functional integral equation like (3), we expect the solution to follow the solutions of

$$\psi(t) = f(t, \psi(t)) \tag{9}$$

and the boundedness of ψ depends on that of $f(t,0)$, as we will see.

We now consider equation (3) in R^n with the Euclidean norm $|\cdot|$

$$x(t) = f(t, x(t)) + \int_0^t u(t, s, x(s))ds, \quad t \geq 0 \tag{3*}$$

where $x(t) \in R^n$ and assume that:

(**i***) $f : R_+ \times R^n \to R^n$ is continuous and there exist a continuous function $k : R_+ \to [0, 1]$ with $0 \leq k(t) < 1$ for $t > 0$ and a constant $x_0 \in R^n$ such that $x_0 = f(0, x_0)$ and

$$\lim_{t \to 0^+} (1 - k(t))^{-1}(f(t, x_0) - f(0, x_0)) = 0,$$

(**ii***) for each $t \in R_+$ and $x, y \in R^n$

$$|f(t, x) - f(t, y)| \leq k(t)|x - y|,$$

(**iii***) $u : R_+ \times R_+ \times R^n \to R^n$ is continuous and there are continuous functions $a, b : R_+ \to R_+$ such that $|u(t, s, x)| \leq a(t)b(s)$ for all $t, s \in R_+, (s \leq t)$ and all $x \in R^n$ with

$$\lim_{t \to 0^+} \frac{a(t)}{1 - k(t)} \int_0^t b(s)ds = 0$$

and

$$\lim_{t \to \infty} \frac{a(t)}{1 - k(t)} \int_0^t b(s)ds = 0.$$

REMARK 4. The first part of (i*) is a necessary condition for equation (3*) to have a solution. The equation $x = f(0, x)$ may have more than one solution, but one and only one x_0 satisfies (i*). It is clear that (i)-(iii) imply (i*)-(iii*). We do not assume $f(t, 0)$ bounded nor do we require a strict contraction condition on $f(t, x)$ for $t = 0$.

We need the following contraction theorem and are unaware if it is already known.

THEOREM 3. (Generalized Contraction). Suppose f satisfies (i*) and (ii*). Then there is a unique continuous function $\psi : R_+ \to R^n$ satisfying (9). Moreover, $|\psi(t)| \leq |f(t, 0)|/(1 - k(t))$ for $t > 0$.

Proof. For each positive integer n we define $(X_n, \|\cdot\|_n)$ as the Banach space of bounded continuous functions $\psi : [\frac{1}{n}, n] \to R^n$ with the supremum norm. We then define a mapping $S_n : X_n \to X_n$ by

$$(S_n\psi)(t) = f(t, \psi(t)), \quad t \in [\frac{1}{n}, n] \tag{10}$$

for $\psi \in X_n$. This is a contraction with a unique fixed point ψ_n, a continuous continuation of ψ_{n-1}. The continuation of ψ_n as $n \to \infty$ is the required function $\psi(t)$ for $0 < t < \infty$. To show that $\psi(t) \to x_0$ as $t \to 0$, where x_0 is given in (i*), we consider for $t > 0$

$$\begin{aligned}|\psi(t) - x_0| =& |f(t, \psi(t)) - f(0, x_0)| \leq |f(t, \psi(t)) - f(t, x_0)| + |f(t, x_0) - f(0, x_0)| \\ &\leq k(t)|\psi(t) - x_0| + |f(t, x_0) - f(0, x_0)|.\end{aligned}$$

This implies that

$$|\psi(t) - x_0| \leq (1 - k(t))^{-1}|f(t, x_0) - f(0, x_0)| \to 0$$

as $t \to 0$. Thus, if we define $\psi(0) = x_0$, then ψ is continuous on R_+ and satisfies (9). Note that ψ is not necessarily bounded. Note also that

$$|\psi(t) - f(t, 0)| = |f(t, \psi(t)) - f(t, 0)| \leq k(t)|\psi(t)|$$

so that

$$|\psi(t)| \leq \frac{1}{1 - k(t)}|f(t, 0)|$$

for $t > 0$. This completes the proof of Theorem 3.

EXAMPLE 1. Consider the function

$$f(t, x) = e^{-t}g(x) + \nu(t)(e^t - 1)$$

where $g : R^n \to R^n$, $\nu : R_+ \to R^n$ are continuous with $|g(x) - g(y)| \leq |x - y|$ for all $x, y \in R^n$ and there exists $x_0 \in R^n$ such that $x_0 = g(x_0)$ and $\nu(0) = g(x_0)$. If we take $k(t) = e^{-t}$, then (i*) and (ii*) are satisfied, and by Theorem 3, there exists a continuous function $\psi : R_+ \to R^n$ satisfying $\psi(t) = f(t, \psi(t))$ for all $t \in R_+$. For $n = 1$, if we take $g(x) = \sin(x + 1)$, then x_0 is the unique solution of $x = \sin(x + 1)$ on $[0, \pi/2]$.

REMARK 5. It is clear that if R_+ is replaced by a finite interval in (i*) and (ii*), Theorem 3 still holds. Thus, if $k(t) = 1$ occurs at infinitely many points on R_+, we may apply Theorem 3 on each finite interval to obtain the following corollary.

COROLLARY 1. Suppose that

(**i****) $f : R_+ \times R^n \to R^n$ is continuous and there exist a continuous function $k : R_+ \to [0, 1]$ with $0 \leq k(t) < 1$ for $t \notin E$, $E = \{t_n \in R_+ | n = 1, 2, \cdots, t_n < t_{n+1}\}$ with $t_n \to \infty$ as $n \to \infty$ and a sequence $\{x_n\}, x_n \in R^n$, such that $x_n = f(t_n, x_n)$ with

$$\lim_{t \to t_n} (1 - k(t))^{-1}(f(t, x_n) - f(t_n, x_n)) = 0,$$

and (ii*) holds. Then there is a unique continuous $\psi : R_+ \to R^n$ satisfying (9).

EXAMPLE 2. Consider the function

$$f(t,x) = \cos t \sin x + \mu(t) \sin^3 t$$

where $\mu : R_+ \to R$ is continuous. If we take $k(t) = |\cos t|$, $t_n = n\pi$ $(n = 1, 2, \cdots)$, and $x_n = 0$ $(n = 1, 2, \cdots)$, then conditions (i**) and (ii*) are satisfied, and by Corollary 1, there exists a unique continuous $\psi : R_+ \to R$ such that $\psi(t) = f(t, \psi(t))$ for all $t \in R_+$.

We need Krasnoselskii's theorem (See Smart [5; p. 31]) in a more general form.

THEOREM 4 . ([3]) Let M be a closed, convex, and nonempty subset of a Banach space $(S, \|\cdot\|)$. Suppose that $A : M \to S$ and $B : S \to S$ such that:
(**i**) B is a contraction with constant $\alpha < 1$,
(**ii**) A is continuous, AM resides in a compact subset of S,
(**iii**) $[x = Bx + Ay, y \in M] \Rightarrow x \in M$.
Then there is a $y \in M$ with $Ay + By = y$.

We use this theorem to prove that every solution of (3*) is asymptotically stable under conditions (i*)-(iii*) without boundedness of $f(t, 0)$. Every solution converges to ψ.

The following compactness result is used.

THEOREM 5. ([4; pp. 79-80]) (Ascoli-type) Let $q : R_+ \to R_+$ be a continuous function such that $q(t) \to 0$ as $t \to \infty$. If $\{\phi_k(t)\}$ is an equicontinuous sequence of R^n-valued functions on R_+ with $|\phi_k(t)| \le q(t)$ for $t \in R_+$, then there is a subsequence that converges uniformly on R_+ to a continuous function $\phi(t)$ with $|\phi(t)| \le q(t)$ for $t \in R_+$, where $|\cdot|$ denotes the Euclidean norm on R^n.

THEOREM 6. Suppose that (i*)-(iii*) hold. Then Equation (3*) has at least one solution. Every solution of (3*) is asymptotically stable and converges to ψ in (9).

Proof. We will use Theorem 3. To arrive at our mapping set we write (3*) as

$$x = y + \psi(t) = f(t, y + \psi(t)) + \int_0^t u(t, s, y(s) + \psi(s))ds$$

or, since $\psi(t) = f(t, \psi(t))$,

$$y = f(t, y + \psi(t)) - f(t, \psi(t)) + \int_0^t u(t, s, y(s) + \psi(s))ds. \tag{11}$$

Our objective is to obtain a solution $y(t)$ as a bounded continuous function; then $x = y + \psi(t)$ is bounded if and only if ψ is bounded.

For X_n given in the proof of Theorem 3, define

$$M_n = \{\phi \in X_n \big| |\phi(t)| \leq q(t)\}, \quad q(t) = \frac{a(t)}{1-k(t)} \int_0^t b(s)ds \tag{12}$$

with $q(0) = 0$ and define $P : X_n \to X_n$ by

$$\begin{aligned}(P\phi)(t) &= f(t, \phi(t) + \psi(t)) - f(t, \psi(t)) + \int_{\frac{1}{n}}^t u(t, s, \phi(s) + \psi(s))ds \\ &=: (B\phi)(t) + (A\phi)(t)\end{aligned} \tag{13}$$

where the order is preserved in that A represents the integral and B the contraction.

LEMMA 1. P is a continuous map of X_n into X_n, B is a contraction, A maps M_n into a compact subset of M_n.

Proof. From (13) we see that ϕ continuous implies $P\phi$ continuous. For each $\phi \in M_n$,

$$|(A\phi)(t)| = \Big|\int_{\frac{1}{n}}^t u(t, s, \phi(s) + \psi(s))ds\Big| \leq a(t) \int_0^t b(s)ds \leq q(t) \tag{14}$$

so that $A : M_n \to M_n$. Clearly, B is a contraction.

To show that A is continuous on M_n, let $\epsilon > 0$ be given. We will find a $\delta > 0$ such that $[\phi, \eta \in M_n,\ \|\phi - \eta\|_n < \delta, \frac{1}{n} \leq t \leq n]$ imply $|(A\phi)(t) - (A\eta)(t)| < \epsilon$. Define

$$H := \sup_{0 \leq t \leq n} q(t) + \sup_{0 \leq t \leq n} |\psi(t)|$$

so that $\|\phi\|_n \leq H$ and $\|\eta\|_n \leq H$. Since $u = u(t, s, y)$ is uniformly continuous on $\frac{1}{n} \leq s \leq t \leq n,\ |y| \leq H$, we find $\delta > 0$ such that $[\frac{1}{n} \leq s \leq t \leq n,\ |\phi(s) - \eta(s)| < \delta]$ imply

$$|u(t, s, \phi(s) + \psi(s)) - u(t, s, \eta(s) + \psi(s))| < \frac{\epsilon}{n}.$$

Thus,

$$|(A\phi)(t) - (A\eta)(t)| \leq \int_{\frac{1}{n}}^t |u(t, s, \phi(s) + \psi(s)) - u(t, s, \eta(s) + \psi(s))|ds < \epsilon$$

for all $t \in [\frac{1}{n}, n]$, and hence A is continuous on M_n.

We want to show that AM_n is in a compact subset of M_n. For $y \in M_n$, we have

$$(Ay)(t) = \int_{\frac{1}{n}}^{t} u(t,s,y(s)+\psi(s))ds$$

so that AM_n is uniformly bounded. It is a standard argument [2; p. 43] to show that is an equicontinuous set. Thus, A maps M_n into a compact subset of M_n.

LEMMA 2. P has a fixed point $\phi_n \in M_n$.

Proof. For fixed $\eta \in M_n$, consider the mapping $D : X_n \to X_n$

$$(D\phi)(t) = (B\phi)(t) + (A\eta)(t).$$

If $D\phi = \phi$, then

$$|\phi(t)| \le k(t)|\phi(t)| + a(t)\int_0^t b(s)ds,$$

from which it follows that $\phi \in M_n$ and P has a fixed point in M_n by Theorem 4.

LEMMA 3. Equation (11) has a solution $\phi \in BC(R_+, R^n)$ satisfying

$$|\phi(t)| \le q(t), \quad t > 0 \tag{15}$$

where $q(t)$ is defined in (12).

Proof. Let ϕ_n be a fixed point of P in (13) on M_n. We have

$$\phi_n(t) = f(t,\phi_n(t)+\psi(t)) - f(t,\psi(t)) + \int_{\frac{1}{n}}^{t} u(t,s,\phi_n(s)+\psi(s))ds.$$

so that

$$|\phi_n(t)| \le k(t)|\phi_n(t)| + a(t)\int_0^t b(s)ds$$

and

$$|\phi_n(t))| \le q(t) \le \sup_{\tau>0} q(\tau) =: \gamma \tag{16}$$

for $1/n \le t \le n$. We may extend the domain of ϕ_n so that it is continuous on R_+ and satisfies (16) for $t \ge 0$. Thus, $\{\phi_n\}$ is a uniformly bounded sequence on R_+ with $\|\phi_n\| \le \gamma$ and $\phi_n(0) = 0$ for all n.

Now, we show that $\{\phi_n\}$ is equi-continuous on R_+. For any $\varepsilon > 0$ and $t_0 \in (0, \infty)$, find $n_0 > 0$ such that $t_0 \in (\frac{1}{n_0}, n_0)$. We first choose $\delta_1 = \min\{\frac{1}{2}(t_0 - \frac{1}{n_0}), \frac{1}{2}(n_0 - t_0)\}$. If $|t - t_0| < \delta_1$, then $t \in [\frac{1}{n_0}, n_0] \subset [\frac{1}{n}, n]$ for $n \geq n_0$ and

$$\begin{aligned}
|\phi_n(t) - \phi_n(t_0)| \leq & |f(t, \phi_n(t) + \psi(t)) - f(t_0, \phi_n(t_0) + \psi(t_0))| + |f(t, \psi(t)) - f(t_0, \psi(t_0))| \\
& + |\int_{\frac{1}{n}}^{t} u(t, s, \phi_n(s) + \psi(s))ds - \int_{\frac{1}{n}}^{t_0} u(t_0, s, \phi_n(s) + \psi(s))ds| \\
\leq & |f(t, \phi_n(t) + \psi(t)) - f(t, \phi_n(t_0) + \psi(t_0))| \\
& + |f(t, \phi_n(t_0) + \psi(t_0)) - f(t_0, \phi_n(t_0) + \psi(t_0))| + |f(t, \psi(t)) - f(t_0, \psi(t_0))| \\
& + |\int_{\frac{1}{n}}^{t} u(t, s, \phi_n(s) + \psi(s))ds - \int_{\frac{1}{n}}^{t_0} u(t_0, s, \phi_n(s) + \psi(s))ds| \\
=: & |f(t, \phi_n(t) + \psi(t)) - f(t, \phi_n(t_0) + \psi(t_0))| + Q_n(t, t_0).
\end{aligned}$$

Let

$$k_0 = \sup\{k(t) | t_0 - \delta_1 \leq t \leq t_0 + \delta_1\}$$

and

$$H_1 = \gamma + \sup\{|\psi(t)| | 0 \leq t \leq t_0 + \delta_1\}$$

where γ is given in (16). Since f and u are uniformly continuous on $[0, t_0 + \delta_1] \times [0, t_0 + \delta_1] \times \{x \in R^n | |x| \leq H_1\}$, there is a $\delta_2 > 0$ such that $|t - t_0| < \delta_2$ implies

$$|\psi(t) - \psi(t_0)| + Q_n(t, t_0) < \varepsilon(1 - k_0)$$

for all $n \geq n_0$. Define $\delta_3 = \min\{\delta_1, \delta_2\}$. If $|t - t_0| < \delta_3$, then

$$\begin{aligned}
|\phi_n(t) - \phi_n(t_0)| &\leq |f(t, \phi_n(t) + \psi(t)) - f(t, \phi_n(t_0) + \psi(t_0))| + Q_n(t, t_0) \\
&\leq k(t)|\phi_n(t) - \phi_n(t_0)| + k(t)|\psi(t) - \psi(t_0)| + Q_n(t, t_0) \\
&\leq k_0|\phi_n(t) - \phi_n(t_0)| + \varepsilon(1 - k_0).
\end{aligned}$$

This implies $|\phi_n(t) - \phi_n(t_0)| < \varepsilon$ whenever $|t - t_0| < \delta_3$ and $n \geq n_0$. Since $\phi_k, k = 1, 2, \cdots, n_0$, are continuous at t_0, there exists $\delta_4 > 0$ such that $|t - t_0| < \delta_4$ implies $|\phi_k(t) - \phi_k(t_0)| < \epsilon$ for all $k \leq n_0$. Thus, if we choose $\delta = \min\{\delta_3, \delta_4\}$, then $|\phi_n(t) - \phi_n(t_0)| < \varepsilon$ whenever $|t - t_0| < \delta$. The equicontinuity of $\{\phi_n\}$ at $t_0 = 0$ follows from the fact that $\phi_n(0) = 0$ and $|\phi_n(t)| \leq q(t) \to 0$ as $t \to 0$ by (iii*). Therefore, $\{\phi_n\}$ is an equicontinuous sequence satisfying $|\phi_n(t)| \leq q(t)$. By Theorem 5, $\{\phi_n\}$ converges uniformly to a continuous function ϕ in R_+. It is clear that ϕ is a solution of (11) and satisfies (15).

Finally, let $x(t) = \phi(t) + \psi(t)$. Then x is a solution of (3*). If y_1 and y_2 are any two solutions of (3*), then $y_i(t) = \phi_i(t) + \psi(t)$ with ϕ_i satisfying (15) for $i = 1, 2$ and

$$|y_1(t) - y_2(t)| \leq |\phi_1(t) - \phi_2(t)| \leq 2q(t) \to 0, \ t \to \infty.$$

Thus, every solution of (3^*) is asymptotically stable and converges to ψ as $t \to \infty$.

EXAMPLE 3. Consider the scalar equation

$$x(t) = e^{-t}g(x) + \nu(t)(e^t - 1) + \int_0^t \beta(t,s,x(s))\frac{x^{1/3}(s)}{1 + |x^{1/3}(s)|}ds \tag{17}$$

where $g(x)$ and $\nu(t)$ are given in Example 1 with $n = 1$, and $\beta : R_+ \times R_+ \times R \to R$ is continuous with $|\beta(t,s,x)| \leq 2t(1+s)/(1+t)^4$ for $t \in R_+$. We can verify that (i*)-(iii*) hold, and by Theorem 6, every solution of (17) is asymptotically stable.

REMARK 5. In Theorem 2 the stability came from Theorem 1 or a look at the proof. Here, the stability is an integral part of the proof since any solution of (11) satisfies (15). Any solution of (11) tends to zero, so any solution of (3^*) approaches ψ: the integral in (3^*) is a "harmless perturbation" of the functional equation $x(t) = f(t, x(t))$.

4. Stability and Nonuniqueness. We have $\psi(t) = f(t, \psi(t))$ and so any solution of (3^*) satisfies $x(0) = \psi(0)$. If solutions are uniquely determined by the initial value then there is only one solution and the problem does seem uninteresting. Banaś and Rszepka [1] mention nothing about uniqueness and their two examples have only unique solutions. But the significance of this study centers squarely on nonuniqueness. To say that a solution of (3^*) is asymptotically stable is to say that if two solutions exist through the unique point $x(0) = \psi(0)$ then the maximum distance between them is

$$\frac{2a(t)\int_0^t b(s)ds}{1 - k(t)}$$

and that this distance tends to zero as $t \to \infty$. In practical terms, that bound can be almost as good as uniqueness.

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