FIXED POINTS AND PROBLEMS IN STABILITY THEORY FOR ORDINARY AND FUNCTIONAL DIFFERENTIAL EQUATIONS

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ABSTRACT. In this paper we begin a study of stability theory for ordinary and functional differential equations by means of fixed point theory. The paper is motivated by a number of difficulties encountered in the study of stability by means of Liapunov's direct method. We notice that these difficulties frequently vanish when we apply fixed point theory. This study is mainly restricted to contraction mappings.

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1. INTRODUCTION

Several problems in differential equations have come to light, mainly in the study of Liapunov's direct method. In a typical problem we have a system of functional differential equations

$$x' = F(t, x_t), \tag{1.1}$$

together with a Liapunov functional $V(t, x_t)$ and continuous increasing functions W_i, satisfying a relation

$$W_1(|x(t)|) \le V(t, x_t) \tag{1.2}$$

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and the derivative of V along a solution of the differential equation satisfying

$$V'(t, x_t) \leq -p(t)W_2(|x(t)|) \tag{1.3}$$

with $p(t) \geq 0$. The objective is to integrate V' and show that if the solution does not go to zero, then V tends to $-\infty$, a contradiction.

Five fundamental difficulties occur. First, it is an art to construct a Liapunov functional satisfying (1.2) and (1.3). Second, conditions need to be placed on $p(t)$ to accomplish the objective and these usually require that p be positive most of the time; these are discussed in [4], [10], and [14], for example. Third, in so much of the work we need to ask that the right hand side of the differential equation be bounded when the space variable is bounded; this is discussed in detail in [1], [3; p. 231], [5], [6], [9] and [13]. Fourth, so often in Liapunov theory, and particularly in Razumikhin theory, when we consider an equation like $x' = a(t)x + b(t)x(t-h)$ there arises a pointwise comparison between a and b; such discussions are found in [3; p. 265], [5], [6], [10], [18], for example. Yet differential equations have solutions expressed as integral equations and we investigate if the comparison should not be averaged instead of being pointwise.

This paper is dedicated to a collection of examples in which these problems arise. We show that all of them can frequently be avoided using fixed point theory, particularly in conjunction with Liapunov theory.

In Part I we consider problems which are half linear and are studied in part by a variation of parameters formula. That formula enables us to set up proper fixed point mappings in such a simple way. Part II considers fully nonlinear problems which require ad hoc techniques for the mappings. Liu [12] considers many nonlinear variation of parameter problems which might be very useful in that connection, but we have not studied that.

Stability definitions may be found in [3], [7], and [17], for example.

PART I: HALF-LINEAR EXAMPLES

2. AN ORDINARY DIFFERENTIAL EQUATION

One of the enduring problems in stability theory is to determine stability properties of the zero solution of equations sharing properties of

$$x' = x^3 - [t/2 + t\sin^n t]x =: f(t, x) \tag{2.1}$$

where $n \geq 1$ is an odd positive integer.

(i) The linear part is asymptotically stable (AS), but not UAS. Hence, AS of the zero solution of (2.1) does not follow from that of the linear part.

(ii) Concerning Liapunov's direct method:

a) For each positive $\epsilon < 1$ there exist sequences t_n and $s_n \to \infty, t_n < s_n < t_{n+1}$ and $f(t_n, \epsilon) < 0, f(s_n, \epsilon/2) > 0$.

b) $f(t, x)$ is not bounded for x bounded.

These are the properties that keep us from using Liapunov functions which are not decrescent. We now show this may not be a difficulty when we use fixed point techniques.

By the variation of parameters formula we write (2.1) as

$$x(t) = x_0 e^{-t^2/4} + \int_0^t e^{(1/4)(-t^2+s^2)}[-sx(s)\sin^n s + x^3(s)]ds. \tag{2.2}$$

Notice that

$$\int_0^t e^{(1/4)(-t^2+s^2)} s ds = 2e^{(1/4)(-t^2)}[e^{(1/4)t^2} - 1] \leq 2.$$

Hence, if $0 < \alpha < 1$, we can find an n so large that

$$\int_0^t e^{(1/4)(-t^2+s^2)}|s \sin^n s| ds \leq \alpha. \tag{2.3}$$

Let $a > 0$, $\|.\|$ be the supremum norm, and let

$$S = \{\phi : [0, \infty) \to R | \phi(0) = x_0, \|\phi\| \leq a, \phi \in C, \phi(t) \to 0 \text{ as } t \to \infty\}. \tag{2.4}$$

The set S is complete. First, it is contained in the Banach space of bounded continuous functions with the supremum norm; thus, any Cauchy sequence ϕ_n has a continuous limit ϕ. As the $\phi_n(t) \to 0$ as $t \to \infty$ we can prove that $\phi(t) \to 0$ and $\|\phi\| \leq a$.

Define $P : S \to S$ by

$$(P\phi)(t) = x_0 e^{-(1/4)t^2} + \int_0^t e^{(1/4)(-t^2+s^2)}[-s\phi(s)\sin^n s + \phi^3(s)]ds. \tag{2.5}$$

THEOREM 2.1. If (2.3) holds then the zero solution of (2.1) is asymptotically stable.

Proof. We will give the details for $t_0 = 0$, but general t_0 offers no new difficulties.

LEMMA 1. There exists $a > 0$ such that $P : S \to S$; that is, $(P\phi)(t)$ is continuous, $(P\phi)(t) \to 0$ as $t \to \infty$, $|(P\phi)(t)| \leq a$, $(P\phi)(0) = x_0$.

Proof. (i) Clearly, $(P\phi)(t)$ is continuous.

Next, we show in two steps that $(P\phi)(t) \to 0$ as $t \to \infty$.

(ii) Note that

$$e^{(1/4)(-t^2+s^2)} = e^{-(1/4)(t-s)(t+s)} \le e^{-(1/4)(t-s)+M}$$

for some $M > 0$. Hence

$$\int_0^t e^M e^{-(1/4)(t-s)} \phi^3(s)ds$$

is the convolution of an L^1 function with a function tending to 0, so the integral tends to zero.

(iii) Now, for each $\epsilon \in (0,1)$, for each $a \in (\epsilon, 1)$, and for each $\phi \in S$, there is a $t_1 > 0$ such that:

a) For $t \ge t_1$ then $|\phi(t)| < \epsilon/4$.

b) And $2ae^{-(3/4)t_1^2} < \epsilon/2$.

Thus, if $t \ge 2t_1$ we have

$$\begin{aligned} e^{-(1/4)t^2} \int_0^t e^{(1/4)s^2} s|\phi(s)|ds =& e^{-(1/4)t^2}[\int_0^{t_1} e^{(1/4)s^2} s|\phi(s)|ds + \int_{t_1}^t e^{(1/4)s^2} s|\phi(s)|ds] \\ \le& 2ae^{-(1/4)t^2} e^{(1/4)t_1^2} + (\epsilon/4) \int_{t_1}^t e^{-(1/4)(t^2-s^2)} sds \\ \le& 2ae^{-(1/4)((2t_1)^2-t_1^2)} + (1/2)\epsilon \\ <& \epsilon. \end{aligned}$$

Hence, $(P\phi)(t) \to 0$ as $t \to \infty$.

(iv) We now determine a so that $\|P\phi\| \le a$. We have

$$\begin{aligned} \|P\phi\| &\le |x_0| + \int_0^t e^{(1/4)(-t^2+s^2)}[|s \sin^n s|a + a^3]ds \\ &\le |x_0| + \alpha a + 2a^3 \end{aligned}$$

for the $\alpha < 1$ of (2.3).

Hence, for a small enough and $|x_0|$ small we have $\|P\phi\| \le a$.

NOTICE. Once we find an a which works, we can obtain smaller and smaller such a. In the $\epsilon - \delta$ definition of stability, for a given ϵ, we will find $a < \epsilon$ and then find x_0 small enough that the mappings work. This value of x_0 will determine δ.

Notice also that for $t \ge 1$

$$\int_0^t e^{(1/4)(-t^2+s^2)} ds \le \int_0^1 e^{(1/4)(-t^2+s^2)} ds + \int_1^t e^{(1/4)(-t^2+s^2)} sds$$

which is bounded (a condition needed for stability).

LEMMA 2. If a is small enough, then P is a contraction.

Proof. We have

$$|(P\phi)(t) - (P\psi)(t)|$$

$$\leq \int_0^t e^{(1/4)(-t^2+s^2)}[|s\sin^n s|\|\phi - \psi\| + \|\phi - \psi\|2(\|\phi\|^2 + \|\psi\|^2)]ds$$

$$\leq \alpha\|\phi - \psi\| + 2\sqrt{\pi}\|\phi - \psi\|(\|\phi\|^2 + \|\psi\|^2)$$

$$\leq \beta\|\phi - \psi\|$$

for some $\beta < 1$ if a is small enough. Here, we used tables to evaluate the integral and obtain $\sqrt{\pi}$.

Now, to finish the proof, given $\epsilon > 0$ we can find $a < \epsilon$ and $\delta > 0$ so that $|x_0| < \delta$ makes Lemma 1 true. Then P has a unique fixed point ϕ and that is the solution tending to zero. Hence, $x = 0$ is asymptotically stable.

3. INDEFINITE a AND UNBOUNDED r IN A DELAY EQUATION

We now give a simple example which so quickly shows how to achieve some of our stated goals. In Hale [7; p. 108] is considered the classical problem

$$x'(t) = -a(t)x(t) - b(t)x(t - r)$$

where a and b are bounded continuous functions,

$$a(t) \geq \delta > 0, |b(t)| \leq \theta\delta, \theta < 1.$$

Using the Liapunov functional

$$V(t, x_t) = (1/2)(x^2(t) + \delta \int_{t-r}^t x^2(s)ds)$$

with the triangle inequality we have

$$V' \leq (\delta/2)(\theta - 1)(x^2(t) + x^2(t - r)).$$

It can now be argued that this yields UAS.

It is severe to ask that a, b be bounded and that $|b(t)|$ is bounded by a all of the time. Lets try to improve that with fixed point theory.

The half-linear equation

$$x'(t) = -a(t)x(t) + b(t)g(x(t - r(t))) \tag{3.1}$$

presents severe challenges when we attempt to show that solutions tend to 0 using Liapunov functionals. We are interested in the case where a can be negative some of the time, a and b are related on average, both a and r' can be unbounded. The problem of boundedness of r' is discussed in Burton-Hatvani [4], Knyazhishche-Shcheglov [10], and Yoshizawa [18], for example. Seifert [15] points out the need for $t - r(t)$ to tend to ∞.

Here, we ask that a, b, and r be continuous, that

$$\int_0^t a(s)ds \to \infty \text{ as } t \to \infty, \tag{3.2}$$

$$\int_0^t e^{-\int_s^t a(u)du}|b(s)|ds \leq \alpha < 1, t \geq 0, \tag{3.3}$$

$$0 \leq r(t), t - r(t) \to \infty \text{ as } t \to \infty, \tag{3.4}$$

there is an $L > 0$ so that if $|x|$, $|y| \leq L$ then

$$g(0) = 0 \text{ and } |g(x) - g(y)| \leq |x - y|. \tag{3.5}$$

THEOREM 3.1. If (3.2)-(3.5) hold, then every solution of (3.1) with small continuous initial function tends to 0 as $t \to \infty$. Moreover, the zero solution is stable at $t_0 = 0$.

Proof. For the α and L, find $\delta > 0$ with $\delta + \alpha L \leq L$. Let $\psi : (-\infty, 0] \to R$ be a given continuous function with $|\psi(t)| < \delta$ and let

$$S = \{\phi : R \to R | \|\phi\| \leq L, \phi(t) = \psi(t) \text{ if } t \leq 0, \phi(t) \to 0 \text{ as } t \to \infty, \phi \in C\}$$

where $\|\cdot\|$ is the supremum norm.

Define $P : S \to S$ by

$$(P\phi)(t) = \psi(t) \text{ if } t \leq 0$$

and

$$(P\phi)(t) = e^{-\int_0^t a(s)ds}\psi(0) + \int_0^t e^{-\int_s^t a(u)du}b(s)g(\phi(s - r(s)))ds, t \geq 0.$$

Clearly, $P\phi \in C$. We now show that $(P\phi)(t) \to 0$ as $t \to \infty$. Let $\phi \in S$ and $\epsilon > 0$ be given. Then $\|\phi\| \leq L$, there exists $t_1 > 0$ with $|\phi(t - r(t))| < \epsilon$ if $t \geq t_1$, and there exists $t_2 > t_1$ such that $t > t_2$ implies that $e^{-\int_{t_1}^t a(u)du} < \epsilon/(L\alpha)$.

Then $t > t_2$ implies that

$$|\int_0^t e^{-\int_s^t a(u)du}b(s)g(\phi(s - r(s)))ds|$$

$$
\begin{aligned}
&\leq \int_0^{t_1} e^{-\int_s^t a(u)du}|b(s)|Lds + \int_{t_1}^t e^{-\int_s^t a(u)du}|b(s)|\epsilon ds \\
&\leq e^{-\int_{t_1}^t a(u)du}\int_0^{t_1} e^{-\int_s^{t_1} a(u)du}|b(s)|Lds + \alpha\epsilon \\
&\leq \alpha L e^{-\int_{t_1}^t a(u)du} + \alpha\epsilon \\
&\leq \epsilon + \alpha\epsilon.
\end{aligned}
$$

To see that P is a contraction under the supremum norm, if $\phi, \eta \in S$, then

$$
|(P\phi)(t) - (P\eta)(t)| \leq \int_0^t e^{-\int_s^t a(u)du}|b(s)|\|\phi - \eta\|ds \leq \alpha\|\phi - \eta\|
$$

with $\alpha < 1$ by (3.3).

Hence, for each such initial function, P has a unique fixed point in S which solves (3.1) and tends to 0.

To get stability for solutions starting at $t_0 = 0$, let $\epsilon > 0$ be given and do the above work for $L = \epsilon$.

4. RELATIONS BETWEEN a, b, AND r

Consider the scalar equation

$$
x' = -a(t)x + \int_{t-r(t)}^t b(t,s)g(x(s))ds \tag{4.1}
$$

where $0 \leq r(t) \leq r_0$. We suppose that

$$
\text{there exists } \alpha < 1 \text{ with } \int_0^t e^{-\int_s^t a(u)du}\int_{s-r(s)}^s |b(s,u)|duds \leq \alpha. \tag{4.2}
$$

Also, for each $\epsilon > 0$ there exist $t_1 > 0$ and $T > 0$ such that $t_2 \geq t_1$ and $t \geq t_2 + T$ imply that

$$
e^{-\int_{t_2}^t a(s)ds} < \epsilon \text{ and } e^{-\int_0^t a(s)ds} \to 0 \text{ as } t \to \infty. \tag{4.3}
$$

Finally, there is an $L > 0$ such that $|x|$, $|y| \leq L$ imply that

$$
|g(x) - g(y)| \leq |x - y| \text{ and } g(0) = 0. \tag{4.4}
$$

THEOREM 4.1. If (4.2) and (4.3) hold then the zero solution of (4.1) is asymptotically stable at $t_0 = 0$.

Proof. For the L and α find $\delta > 0$ so that $\delta + \alpha L \leq L$. Let $\psi : [-r_0, 0] \to R$ be a given continuous initial function with $|\psi(t)| < \delta$ and define

$$S = \{\phi : [-r_0, \infty) \to R | \|\phi\| \leq L, \phi \in C, \phi_0 = \psi, \phi(t) \to 0 \text{ as } t \to \infty\}$$

where $\|\cdot\|$ denotes the supremum norm.

Define $P : S \to S$ by

$$(P\phi)(t) = \psi(t) \text{ if } -r_0 \leq t \leq 0$$

and

$$(P\phi)(t) = e^{-\int_0^t a(s)ds}\psi(0) + \int_0^t e^{-\int_s^t a(u)du} \int_{s-r(s)}^s b(s,u)g(\phi(u))duds, t \geq 0.$$

Now, let $\epsilon > 0$ and $\phi \in S$ be given. Then $\|\phi\| \leq L$ and there exists $t_1 > 0$ such that $t \geq t_1 - r_0$ implies that $|\phi(t)| < \epsilon$, while $t_2 \geq t_1$ and $t \geq t_2 + T$ imply that $e^{-\int_{t_2}^t a(s)ds} < \epsilon$. Thus, by the second part of (4.3) we may suppose t_1 so large that $e^{-\int_0^{t_1} a(s)ds} < \epsilon$. Hence, $t \geq t_2 + T$ implies that

$$\begin{aligned}
&|(P\phi)(t)| \\
&\leq \delta\epsilon + \int_0^{t_2} e^{-\int_s^{t_2} a(u)du} e^{-\int_{t_2}^t a(u)du} \int_{s-r(s)}^s |b(s,u)| L duds \\
&\quad + \int_{t_2}^t e^{-\int_s^t a(u)du} \int_{s-r(s)}^s |b(s,u)|\epsilon ds \\
&\leq \delta\epsilon + \epsilon\alpha L + \epsilon\alpha.
\end{aligned}$$

Thus, $(P\phi)(t) \to 0$ as $t \to \infty$. Also, $\|\phi\| \leq L$ implies $\|P\phi\| \leq L$ by choice of δ.

Finally,

$$\begin{aligned}
|(P\phi)(t) - (P\eta)(t)| &\leq \int_0^t e^{-\int_s^t a(u)du} \int_{s-r(s)}^s |b(s,u)| duds \|\phi - \eta\| \\
&\leq \alpha \|\phi - \eta\|.
\end{aligned}$$

Hence, P has a unique fixed point in S.

5. A SCALAR NEUTRAL EQUATION

We now consider problems in which the ode part is not necessarily stable and the delay part will be used to stabilize it. Such work with Liapunov functions is found, for example, in Burton [2; p. 139]. The idea is to convert to a neutral equation. Here, the size of the delay will certainly play a role.

The reader can confirm that the work we do here also carries through on a nonlinear equation

$$x'(t) = -a(t)x(t) + b(t)x(t-r) + h(t)g(x(t-r))$$

where g satisfies a local Lipschitz condition, as in previous sections. But in order to not obscure the main part, we consider the scalar equation

$$x'(t) = -a(t)x(t) + b(t)x(t-r)$$

$$= [-a(t) + b(t+r)]x(t) - (d/dt)\int_{t-r}^{t} b(s+r)x(s)ds$$

$$=: A(t)x(t) - (d/dt)\int_{t-r}^{t} b(s+r)x(s)ds. \tag{5.1}$$

It is assumed that

$$\int_0^t [-a(s) + b(s+r)]ds \to -\infty \text{ as } t \to \infty, \tag{5.2}$$

$$\int_{t-r}^{t} |b(u+r)|du + \int_0^t |A(s)e^{\int_s^t A(u)du}| \int_{s-r}^{s} |b(u+r)|duds \le \alpha < 1, \tag{5.3}$$

and that whenever $\phi(t) \to 0$ as $t \to \infty$, then

$$\int_0^t A(s)e^{\int_s^t A(u)du} \int_{s-r}^{s} b(u+r)\phi(u)duds \to 0 \text{ as } t \to \infty. \tag{5.4}$$

EXAMPLE 5.1. Let $a(t) = 0, b(t) = b < 0$ a constant. Then (5.2) and (5.4) are immediately satisfied, while (5.3) needs $-2br < 1$.

THEOREM 5.1. If (5.2)-(5.4) hold, then for every continuous $\psi : [-r, 0] \to R$, the solution $x(t, 0, \psi) \to 0$ as $t \to \infty$.

Proof. Let

$$S = \{\phi : [-r, \infty) \to R | \phi \in C, \phi_0 = \psi, \phi(t) \to 0 \text{ as } t \to \infty\}.$$

We can integrate by parts and write the solution of (6.1) as

$$x(t) = e^{\int_0^t A(s)ds}\psi(0) - \int_0^t e^{\int_s^t A(u)du}(d/ds)\int_{s-r}^{s} b(u+r)x(u)duds$$

$$= e^{\int_0^t A(s)ds}\psi(0) - e^{\int_s^t A(u)du}\int_{s-r}^{s} b(u+r)x(u)du|_0^t$$

$$-\int_0^t A(s)e^{\int_s^t A(u)du}\int_{s-r}^s b(u+r)x(u)duds.$$

$$= e^{\int_0^t A(s)ds}\psi(0) - \int_{t-r}^t b(u+r)x(u)du + e^{\int_0^t A(u)du}\int_{-r}^0 b(u+r)\psi(u)du$$

$$-\int_0^t A(s)e^{\int_s^t A(u)du}\int_{s-r}^s b(u+r)x(u)duds.$$

Now, define $P : S \to S$ by $(P\phi)(t) = \psi(t)$ if $-r \le t \le 0$ and for $t \ge 0$ then

$$(P\phi)(t) = e^{\int_0^t A(s)ds}\psi(0) - \int_{t-r}^t b(u+r)\phi(u)du + e^{\int_0^t A(u)du}\int_{-r}^0 b(u+r)\psi(u)du$$

$$-\int_0^t A(s)e^{\int_s^t A(u)du}\int_{s-r}^s b(u+r)\phi(u)duds.$$

By (5.2) and (5.4), $(P\phi)(t) \to 0$ as $t \to \infty$. To see that P is a contraction, if $\phi, \eta \in S$ then

$$|(P\phi)(t) - (P\eta)(t)| \le \int_{t-r}^t |b(u+r)|\|\phi - \eta\|du$$

$$+\int_0^t |A(s)e^{\int_s^t A(u)du}|\int_{s-r}^s |b(u+r)|duds\|\phi - \eta\|$$

$$\le \alpha\|\phi - \eta\|$$

by (5.3). Hence, P is a contraction and there is a unique solution tending to zero.

6. AN ENTIRELY DELAYED EQUATION

Here, we consider

$$x' = -b(t)g(x(t-h(t)) \tag{6.1}$$

where $b(t) \ge 0$, $0 \le h(t) \le r_0$, and $1 - h'(t) > 0$ so that the function defined by $t - h(t)$ is strictly increasing and has an inverse $r(t)$. We can write (6.1) as

(6.2)

$$x' = -[b(r(t))/(1-h'(r(t)))]g(x) + (d/dt)\int_{t-h(t)}^t [b(r(s))g(x(s))/(1-h'(r(s))]ds$$

or

$$x' = -c(t)g(x) + (d/dt)\int_{t-h(t)}^t c(s)g(x(s))ds. \tag{6.3}$$

We suppose that

$$g(x) = x + G(x), \lim_{x\to 0}[G(x)/x] = 0, c(t) \ge 0. \tag{6.4}$$

(With considerable care, we can reduce $c(t) \geq 0$.)

For a given initial function $\psi : [-r_0, 0] \to R$, the solution is given by

$$x(t) = e^{-\int_0^t c(s)ds}\psi(0)$$

$$+ \int_0^t e^{-\int_s^t c(u)du}[-c(s)G(x(s)) + (d/ds)\int_{s-h(s)}^s c(u)g(x(u))du]ds.$$

Integrating that derivative by parts will yield

$$x(t) = e^{-\int_0^t c(s)ds}\psi(0) - \int_0^t e^{-\int_s^t c(u)du}c(s)G(x(s))ds$$

$$+ \int_{t-h(t)}^t c(u)g(x(u))du - e^{-\int_0^t c(u)du}\int_{-h(0)}^0 c(u)g(x(u))du$$

$$- \int_0^t c(s)e^{-\int_s^t c(u)du}\int_{s-h(s)}^s c(u)g(x(u))duds.$$

We suppose that there is a $\delta > 0$ and $\gamma > 0$ so that if

$$g^* = \max_{|x| \leq \gamma} |g(x)|, G^* = \max_{|x| \leq \gamma} |G(x)| \tag{6.5}$$

(this is not needed in the linear case) then

$$\delta + G^* + g^* \int_{t-h(t)}^t c(u)du + g^* \int_{-h(0)}^0 c(u)du$$

$$+g^* \int_0^t e^{-\int_s^t c(u)du}c(s)\int_{s-h(s)}^s c(u)duds < \gamma. \tag{6.6}$$

(This, too, is not needed in the linear case.)

EXAMPLE 6.1. If $g(x) = x$ then $g^* = \gamma$ and $G^* = 0$. If $h = 1$, then the inverse of $t - 1$ is $r(t) = t + 1$. We then need

$$\delta + \gamma \int_{t-1}^t c(u)du + \gamma \int_{-1}^0 c(u)du + \gamma \int_0^t e^{-\int_s^t c(u)du}c(s)\int_{s-1}^s c(u)duds < \gamma.$$

If we take c to be a positive constant, this reduces to

$$c < 1/3.$$

Two more conditions are needed. First, there are $\lambda \geq 0, \mu > 0$ such that

$$|x|, |y| \leq \gamma \text{ imply } |g(x) - g(y)| \leq \mu|x - y|, |G(x) - G(y)| \leq \lambda|x - y|. \tag{6.7}$$

Also,

$$\lambda + \mu \int_{t-h(t)}^{t} c(u)du + \mu \int_0^t e^{-\int_s^t c(u)du} c(s) \int_{s-h(s)}^{s} c(u)duds \le \alpha < 1. \tag{6.8}$$

EXAMPLE 6.2. If c is constant, $g(x) = x$, and $h = 1$ then $\lambda = 0, \mu = 1$, and (6.8) is satisfied if $c < 1/2$.

THEOREM 6.1. Let (6.4)-(6.8) hold and $\int_0^t c(s)ds \to \infty$ as $t \to \infty$. Then $\|\psi\| < \delta$ implies that $|x(t, 0, \psi)| \to 0$ as $t \to \infty$.

Proof. For the given ψ with $\|\psi\| < \delta$ define

$$S = \{\phi : [-r_0, \infty) \to R | \phi \in C, \phi(t) \to 0, \phi_0 = \psi, \|\phi\| \le \gamma\}.$$

Then define $P : S \to S$ by $\phi \in S$ implies that $(P\phi)(t) = \psi(t)$ if $-r \le t \le 0$ and for $t \ge 0$ then

$$(P\phi)(t) = e^{-\int_0^t c(s)ds}\psi(0) - \int_0^t e^{-\int_s^t c(u)du} c(s)G(\phi(s))ds$$

$$+ \int_{t-h(t)}^{t} c(u)g(\phi(u))du - e^{-\int_0^t c(u)du} \int_{-h(0)}^{0} c(u)g(\psi(u))du$$

$$- \int_0^t e^{-\int_s^t c(u)du} c(s) \int_{s-h(s)}^{s} c(u)g(\phi(u))duds.$$

It is easy to argue that if $\phi(t) \to 0$ as $t \to \infty$ and (6.6) holds then $(P\phi)(t) \to 0$ as $t \to \infty$. Also, (6.6) shows that $\phi \in S$ implies that $\|P\phi\| \le \gamma$.

To see that P is a contraction,

$$|(P\phi)(t) - (P\eta)(t)| \le \lambda\|\phi - \eta\| + \mu \int_{t-h(t)}^{t} c(u)du\|\phi - \eta\|$$

$$+\mu\|\phi - \eta\| \int_0^t e^{-\int_s^t c(u)du} c(s) \int_{s-h(s)}^{s} c(u)duds$$

$$\le \alpha\|\phi - \eta\|$$

by (6.8).

REMARK 6.1. This problem has been widely discussed and much is known about it. Note first that our conditions can hold even when $b(t)$ is zero over arbitrarily long time periods, so long as it has an infinite integral. By contrast, standard results usually ask that b be integrally positive in some sense. On the other hand, some of the best known results ask that

$$b_s := \sup_{t \ge -h} \int_t^{t+h} |b(u + h)|du < 3/2.$$

These are discussed in some detail in Knyazhishche-Shcheglov [10] which uses sophisticated Liapunov theory. Their result asks that $b_s \leq \ln 4$, which is much better than ours, but also requires a very complicated integrally positive condition which requires a separate theorem to verify.

Other treatment of (6.1) by Liapunov theory is found in Burton-Hatvani [4; pp. 68 and 79] where it is required that b be integrally positive. Further results are found in Yoneyama [16], for example.

7. A d-DIMENSIONAL NEUTRAL VOLTERRA EQUATION

We actually want solutions to be $L^1[0,\infty)$, but the set S which we have been constructing would not be complete if we asked for such functions. However, we will go about matters in a very indirect manner here.

Consider the d-dimensional system

$$x' = Ax + \int_0^t C(t-s)x(s)ds + \int_0^t D(t-s)x(s)ds \tag{7.1}$$

where

$$C, D, \int_t^\infty C(v)dv \in L^1[0,\infty) \text{ and are continuous.} \tag{7.2}$$

We let
(7.3)

$$Q = A + \int_0^\infty C(v)dv \text{ and assume that all roots of Q have negative real parts.}$$

Then

$$x' = Qx - (d/dt)\int_0^t \int_{t-s}^\infty C(v)dvx(s)ds + \int_0^t D(t-s)x(s)ds \tag{7.4}$$

and

$$x(t) = e^{Qt}x_0 - \int_0^t e^{Q(t-s)}(d/ds)\int_0^s \int_{s-u}^\infty C(v)dvx(u)duds$$

$$+ \int_0^t e^{Q(t-s)} \int_0^s D(s-u)x(u)duds. \tag{7.5}$$

We define S as before (without a norm restriction since (7.1) is linear) and define $P : S \to S$ through (7.5) but integrate by parts to get

$$(P\phi)(t) = e^{Qt}x_0 - e^{Q(t-s)}\int_0^s \int_{s-u}^\infty C(v)dv\phi(u)du\Big|_0^t$$

$$- \int_0^t Qe^{Q(t-s)} \int_0^s \int_{s-u}^{\infty} C(v)dv\phi(u)duds$$

$$+ \int_0^t e^{Q(t-s)} \int_0^s D(s-u)\phi(u)duds$$

or

$$(P\phi)(t) = e^{Qt}x_0 - \int_0^t \int_{t-u}^{\infty} C(v)dv\phi(u)du$$

$$(7.6) \quad - \int_0^t Qe^{Q(t-s)} \int_0^s \int_{s-u}^{\infty} C(v)dv\phi(u)duds + \int_0^t e^{Q(t-s)} \int_0^s D(s-u)\phi(u)duds.$$

Now P will be a contraction if

$$|(P\phi)(t) - (P\eta)(t)| \leq [\int_0^t \int_{t-u}^{\infty} |C(v)|dvdu$$

$$+ \int_0^t |Qe^{Q(t-s)}| \int_0^s |\int_{s-u}^{\infty} C(v)dv|duds$$

$$+ \int_0^t |e^{Q(t-s)}| \int_0^s |D(s-u)|duds]\|\phi - \eta\|$$

$$(7.7) \qquad \leq \alpha\|\phi - \eta\|, \alpha < 1.$$

THEOREM 7.1. If (7.2), (7.3), and (7.7) hold then every solution $x(t, 0, x_0)$ of (1) tends to 0 as $t \to \infty$.

8. A SEARCH FOR NORMS

In this section we approach Section 7 differently. Consider the system

$$z' = Az + \int_0^t B(t-s)z(s)ds,$$

where A is constant, $\int_0^\infty |B(s)|ds < \infty$, and $Z(t)$ is the $n \times n$ matrix of solutions with $Z(0) = I$. It is known (cf. Burton [2; pp. 49-51]) that $z = 0$ is UAS if and only if $Z \in L^1[0, \infty)$.

Question: What then is the appropriate norm for solutions?

1. When solutions are US, the supremum norm seems right.

2. When solutions tend to 0, this should be reflected in the norm or in the space.

3. When $Z \in L^1$, the norm should include $\int_0^\infty |z(s)|ds$.

Thus, we now let

$$S = \{\phi : [0,\infty) \to R^d | \phi \in C, \int_0^\infty |\phi(u)|du < \infty, \phi(t) \to 0\}$$

with norm

$$\|\phi\| = \sup_{0 \le t < \infty} |\phi(t)| + \int_0^\infty |\phi(u)|du =: |\phi|_\infty + |\phi|_1.$$

This space is complete. We now return to the work from the previous problem and study it from a different point of view.

Consider the d-dimensional system

$$x' = Ax + \int_0^t C(t-s)x(s)ds + \int_0^t D(t-s)x(s)ds \tag{8.1}$$

where

$$C, D, \int_t^\infty C(v)dv \in L^1[0,\infty) \text{ and are continuous.} \tag{8.2}$$

We let
(8.3)
$Q = A + \int_0^\infty C(v)dv$ and assume that all roots of Q have negative real parts.

Then exactly as in the last section we write (8.1) as an integral equation, integrate by parts, and define $P : S \to S$ by

$$(P\phi)(t) = e^{Qt}x_0 - \int_0^t \int_{t-u}^\infty C(v)dv\phi(u)du$$

$$(8.6)\quad -\int_0^t Qe^{Q(t-s)} \int_0^s \int_{s-u}^\infty C(v)dv\phi(u)duds + \int_0^t e^{Q(t-s)} \int_0^s D(s-u)\phi(u)duds.$$

REMARK 8.2. Note that if $\phi(t) \to 0$ as $t \to \infty$, then each integral is the convolution of an L^1- function with a function tending to 0. (Do the middle integral twice.) Hence, each integral tends to 0. In exactly the same way, if $\phi \in L^1$ then we have $P\phi \in L^1$.

Now $P : S \to S$ independent of any norm. Under the sup norm, S is complete and P will be a contraction if

$$|(P\phi)(t) - (P\eta)(t)| \le [\int_0^t \int_{t-u}^\infty |C(v)|dvdu$$

$$+\int_0^t |Qe^{Q(t-s)}| \int_0^s |\int_{s-u}^\infty C(v)dv|duds$$

$$+\int_0^t |e^{Q(t-s)}| \int_0^s |D(s-u)|duds]\|\phi-\eta\|$$

$$\leq \alpha\|\phi-\eta\|, \alpha<1. \tag{8.7}$$

And if it is a contraction, then it has a unique fixed point in S and that fixed point is a solution of (8.1) which is integrable.

THEOREM 8.1. If (8.2), (8.3), and (8.7) hold then every solution $x(t,0,x_0)$ of (8.1) is in $L^1[0,\infty)$ and so the zero solution is UAS.

9. ULTIMATE BOUNDEDNESS

Ultimate boundedness using Liapunov functionals has been a very elusive problem. The reader is referred to Yoshizawa [17] for definitions of uniform boundedness (UB) and uniform ultimate boundedness (UUB). Many typical details are found in Burton [3; pp. 272-279]. Very special conditions on the Liapunov functionals are required and Hale [7; p. 139] declines to discuss them because of severe restrictions on the size of the delay.

If we consider

$$x' = -a(t)x + b(t)g(x(t-h)) + f(t) \tag{9.1}$$

with $a(t) \geq |b(t+h)|$ and $|f(t)| \leq m, |g(x)| \leq r|x|$, and $a(t) \geq \alpha > 0$, then the natural Liapunov functional will not work unless a, b, f are bounded. The usual argument goes like this. Define

$$V(t,x_t) = x^2(t) + \int_{t-h}^t |b(s+h)|g^2(x(s))ds$$

so that

$$V' = -2a(t)x^2 + 2b(t)xg(x(t-h)) + 2f(t)x + |b(t+h)|g^2(x) - |b(t)|g^2(x(t-h))$$

$$\leq [-2a(t) + |b(t)| + m^2 + r^2|b(t+h)|]x^2 + f^2(t)/m^2$$

and we need

$$2a(t) \geq |b(t)| + r^2|b(t+h)| + m^2 + \gamma \tag{C}$$

where m, γ are positive constants. Because of the assumed boundedness of b and f we arrive at the system

$$W_1(|x(t)|) \leq V(t,x_t) \leq W_2(|x(t)| + \int_{t-h}^t W_3(|x(s)|)ds)$$

$$V' \leq -W_3(|x(t)|) + K$$

for appropriate $K > 0$ and wedges W_i. Because of the very special relation with W_3 in both the upper bound and in the derivative, this will readily yield UB and UUB (Burton [3; p. 278]).

We now use a contraction to relax (C). In (9.1), suppose that there are positive constants M, L, K, μ with

$$\int_0^t e^{-\int_s^t a(u)du}|f(s)|ds \leq M, \tag{9.2}$$

(Thus, on average f is dominated by a.)

$$\int_0^t e^{-\int_s^t a(u)du}|b(s)|ds \leq L, \tag{9.3}$$

(And on average b is also dominated by a.)

$$|g(x) - g(y)| \leq \mu|x - y|, g(0) = 0, \mu LK + M + 1 < K, \tag{9.4}$$

(This implies that $\mu L < 1$.) and for each $t_1 > 0$ and $\epsilon > 0$ there exists $t_2 > t_1$ such that $t > t_2$ implies that

$$e^{-\int_0^t a(s)ds} < \epsilon \text{ and } e^{-\int_{t_1}^t a(u)du} < \epsilon. \tag{9.5}$$

(Thus, a is positive on average.)

THEOREM 9.1. If (9.2)-(9.5) hold, then every solution of (9.1) satisfies $|x(t, 0, \psi)| < K$ for all large t.

Proof. Let $\psi : [-h, 0] \to R$ be a given continuous initial function and define

$$S = \{\phi : [-h, \infty) \to R | \phi_0 = \psi, \phi \in C, |\phi(t)| < K \text{ for large } t\}.$$

Next define $P : S \to S$ by

$$(P\phi)(t) = \psi(t) \text{ if } t < 0$$

$$(P\phi)(t) = \psi(0)e^{-\int_0^t a(s)ds} + \int_0^t e^{-\int_s^t a(u)du}[b(s)g(\phi(s - h)) + f(s)]ds, t \geq 0.$$

Let $\phi \in S$ be given. Then there exists J such that $\|\phi\| \leq J$ and there exists t_1 such that $t \geq t_1 - h$ implies that $|\phi(t)| \leq K$. Select $\epsilon > 0$ so that $|\phi(0)|\epsilon + L\mu J\epsilon < 1/2$. Next, find t_2 and let $t \geq t_2 > t_1$ so that we will have

$$|(P\phi)(t)| \leq |\phi(0)|e^{-\int_0^t a(s)ds} + M$$

$$+ \int_0^{t_1} e^{-\int_s^{t_1} a(u)du} e^{-\int_{t_1}^t a(u)du}|b(s)|\mu J ds + \int_{t_1}^t e^{-\int_s^t a(u)du}|b(s)|K\mu ds$$

$$\le |\phi(0)|\epsilon + M + L\mu J\epsilon + K\mu L$$
$$< K - (1/2).$$

Thus, $P : S \to S$. In fact, it is mapped into a closed subset of S. We will show in a moment that P is a contraction. Now S may not be complete because a Cauchy sequence may have a limit ϕ with $|\phi(t)| \le K$ for large t. However, S is mapped into a complete subset of itself. To see this, refer to a standard proof of the contraction mapping theorem; if $\phi \in S$, consider the sequence $\{P^k\phi\}$ which we have just shown to be in S and $|(P^k\phi)(t)| < K - (1/2)$ for large t. This is a Cauchy sequence and it is in the Banach space of bounded continuous functions with the supremum norm so it has a limit η and $|\eta(t)| < K - (1/4)$ for large t. That limit is the fixed point.

Now
$$|(P\phi)(t) - (P\eta)(t)| \le \int_0^t e^{-\int_s^t a(u)du}|b(s)|\mu ds\|\phi - \eta\|$$
$$\le \mu L\|\phi - \eta\|,$$
a contraction since $\mu L < 1$.

This example shows that for this half-linear equation, the classical restriction (C) can be significantly improved using fixed point theory.

PART II: NONLINEAR EXAMPLES

The variation of parameters formula does two things for us. It allows us to define a mapping so that the image of a function has the correct initial condition and we can prove that the image tends to zero as $t \to \infty$. One of the classical fully nonlinear examples in Liapunov theory may be found in Hale [7; pp. 117-118] and it concerns

$$(E) \qquad x'(t) = a(t)x^3(t) + b(t)x^3(t-h)$$

with a and b bounded continuous functions satisfying

$$a(t) \le -\delta < 0, |b(t)| < q\delta, 0 < q < 1.$$

An appropriate Liapunov functional is

$$V(t, x_t) = (1/4)x^4(t) + (\delta/2)\int_{t-h}^t x^6(s)ds$$

so that
$$V'(t, x_t) \le (\delta/2)(q-1)(x^6(t) + x^6(t-h)).$$

From these relations and Marachkoff's argument one can prove that the zero solution is uniformly asymptotically stable.

We will now perturb (E) in several different ways and see what can be proved using fixed point theory.

10. A NONLINEAR ODE

Consider the scalar equation

(10.1) $$x'(t) = -a(t)x^3 + g(t, x^2, x)$$

where

$$a(t) \geq 0, \int_0^\infty a(t)dt = \infty, g(t, y, 0) = 0,$$

and

(10.2) $$|g(t, y, x) - g(t, y, w)| \leq b(t)|y||x - w|.$$

Now we come to a recurring problem. We require that for each bounded continuous function $z^2(t)$ with $z^2(t) \geq c$ for some $c > 0$, there is an $\alpha < 1$ with

$$\int_0^t e^{-\int_s^t a(u)z^2(u)du} b(s)z^2(s)ds \leq \alpha, t \geq 0.$$

An obvious sufficient condition is that $a(t) \geq kb(t)$ and $a(t) \geq 0$ for all t and some $k > 1$. But that is much too severe. We would like for a to be zero on long intervals when b is nonzero.

But here we come to a real difficulty. In these fully nonlinear problems we will use the unknown exact solution as part of the mapping. Thus, we need to rely on additional information to ensure that solutions exist on $[t_0, \infty)$. An example of (10.1) is

$$x' = -a(t)x^3 + b(t)x^3.$$

If $a(t) < b(t)$ and $b(t) > 0$ on any interval $[t_0, t_1]$, however short, there are solutions with finite escape time. Hence, it will be necessary to work with particular initial times t_0 for which

(*) $$\int_{t_0}^t [-a(s) + b(s)]ds \leq 0 \text{ for all } t \geq t_0.$$

Obviously, if $a(t) \geq b(t)$ for all t, this would hold for any t_0.

LEMMA 3. Let (*) hold and let $x_0 \in R$. Then $x(t, t_0, x_0)$ is defined for all $t \geq t_0$.

Proof. Let $x(t) = x(t, t_0, x_0)$ be a solution of (10.1) with maximal interval of definition $[t_0, t_1)$. It is known that $t_1 = \infty$ or $\lim_{t \to t_1} |x(t)| = \infty$. Then define a Liapunov function

$$V(x) = |x|$$

so that along the solution with have

$$V'(x(t)) \leq -a(t)|x(t)|^3 + b(t)|x(t)|^3 = [-a(t) + b(t)]V^3.$$

If we separate variables and integrate we obtain

$$-V(t)^{-2}+V(t_0)^{-2}\le 2\int_{t_0}^{t}[-a(s)+b(s)]ds\le 0$$

so that $|x(t)|^3\le|x(t_0)|^3$, a contradiction to the finite escape time.

We now assume that for a given x_0 and a given $c<|x_0|$, there is an $\alpha<1$ such that if $z^2(t)$ is continuous and $x_0^2\ge z^2(t)\ge c$, then

$$\int_{t_0}^{t}e^{-\int_s^t a(u)z^2(u)du}b(s)z^2(s)ds\le\alpha \text{ for } t\ge t_0. \tag{10.3}$$

LEMMA 4. Suppose that if $t_n\to\infty$ and if

$$K_n=\sup_{t\ge t_n}\int_{t_n}^{t}[-a(s)+b(s)]ds,\ \text{then}\ \lim_{n\to\infty}K_n=0. \tag{**}$$

If $x(t)$ is a solution of (10.1) on $[t_0,\infty)$ either $x(t)\to 0$ or there is a $c>0$ with $|x(t)|\ge c$.

Proof. Suppose there is a sequence $t_n\to\infty$, a sequence $s_n>t_n$, and a $c>0$ with $|x(t_n)|\to 0$ and $|x(s_n)|=c$. Rename indices so that $|x(t_n)|<c/2$ for all n. Using the V as in the proof of Lemma 3, and taking

$$k_n=\int_{t_n}^{s_n}[-a(s)+b(s)]ds$$

we have, upon integration of V', the relation

$$-V^{-2}(s_n))+V^{-2}(t_n))\le 2k_n.$$

If some $k_n\le 0$, then $V(s_n)\le V(t_n)$, a contradiction. Hence, $k_n\ge 0$ for all n and $k_n\le K_n\to 0$ as $n\to\infty$ so

$$V^{-2}(t_n))-2k_n\le c^{-2},$$

a contradiction since $V(t_n)\to 0$.

In particular, from this result, for each x_0 there is a solution $x(t,0,x_0)=:z(t)$ defined on $[0,\infty)$. If we strengthen the Lipschitz condition in (10.2) it is unique. We will now outline the method to be used on these problems.

1. We shall assume or prove that there is a t_0 and for each x_0 there is a unique solution $x(t,t_0,x_0)=:z(t)$ on $[t_0,\infty)$, When (*) and (**) hold we have shown in the lemmas how this might be done in a particular problem.

2. Hence, $z(t)$ is the unique solution of

$$x'=-a(t)x^2(t)x+g(t,z^2(t),x),x(t_0)=x_0. \tag{10.4}$$

3. PROBLEM. In what space does $z(t)$ lie? We want to show that it lies in

$$S = \{\phi : [t_0, \infty) \to R | \phi(t_0) = x_0, \phi(t) \to 0 \text{ as } t \to \infty, \phi \in C\}. \tag{10.5}$$

Here, $\| \cdot \|$ will denote the supremum norm.

The unique solution of (10.4) is

$$x(t) = x_0 e^{-\int_{t_0}^t a(s)z^2(s)ds} + \int_{t_0}^t e^{-\int_s^t a(u)z^2(u)du} g(s, z^2(s), x(s))ds. \tag{10.6}$$

4. Define $P : S \to S$ by

$$(P\phi)(t) = x_0 e^{-\int_{t_0}^t a(s)z^2(s)ds} + \int_{t_0}^t e^{-\int_s^t a(u)z^2(u)du} g(s, z^2(s), \phi(s))ds. \tag{10.7}$$

5. If P has a fixed point, it is z, and so $z \in S$ which means that $z(t) \to 0$.

6. In this example, when (*) and (**) hold we know from the lemmas that either:

a) $z(t) \to 0$, so there is nothing to prove, or

b)$|z(t)| \geq c > 0$ so $\int_0^\infty a(t)z^2(t)dt = \infty$.

Thus, we assume that b) holds.

7. Clearly, $(P\phi)(t_0) = x_0$ and $P\phi \in C$. We now show that $(P\phi)(t) \to 0$ as $t \to \infty$ and we take $t_0 = 0$ for brevity.

Let $\epsilon > 0$ and $\phi \in S$ be given and let $c > 0$ be found. Find t_1 so that $|\phi(t)| < \epsilon/2$ if $t \geq t_1$. Then using (10.3) we obtain

$$\int_0^t e^{-\int_s^t a(u)z^2(u)du} |g(s, z^2(s), \phi(s))|ds$$

$$\leq \int_0^{t_1} e^{-\int_s^t a(u)z^2(u)du} b(s)z^2(s)|\phi(s)|ds + \int_{t_1}^t e^{-\int_s^t a(u)z^2(u)du} b(s)z^2(s)|\phi(s)|ds$$

$$\leq e^{-\int_{t_1}^t a(u)z^2(u)du} \int_0^{t_1} e^{-\int_s^{t_1} a(u)z^2(u)du} b(s)z^2(s)\|\phi\|ds$$

$$+(\epsilon/2) \int_0^t e^{-\int_s^t a(u)z^2(u)du} b(s)z^2(s)ds$$

$$\leq \alpha \|\phi\| e^{-\int_{t_1}^t a(u)z^2(u)du} + (\epsilon/2)\alpha.$$

The first term tends to zero as $t \to \infty$ and the second term can be made as small as we please.

8. To see that we have a contraction, for $\phi, \psi \in S$ we have

$$
\begin{aligned}
|(P\phi)(t) - (P\psi)(t)| &\leq \int_{t_0}^{t} e^{-\int_s^t a(u)z^2(u)du} |g(s, z^2(s), \phi(s)) - g(s, z^2(s), \psi(s))| ds \\
&\leq \int_{t_0}^{t} e^{-\int_s^t a(u)z^2(u)du} b(s)z^2(s)|\phi(s) - \psi(s)| ds \\
&\leq \alpha \|\phi - \psi\|.
\end{aligned}
$$

Thus, P does have a fixed point and it is in S.

11. A FIRST NONLINEAR DELAY EQUATION

Let a and b be continuous, $h > 0$, and consider the equation

$$
x'(t) = -a(t)x^3(t) + b(t)x(t-h)x^2(t) \tag{11.1}
$$

This is a difficult problem even if we ask that $a(t) > |b(t)|$ since $b(t)x(t-h)x^2(t)$ can dominate $-a(t)x^3(t)$ along a solution in a neighborhood of zero. But under that condition it is possible to find a mildly unbounded Liapunov function which will show that all solutions can be continued for all future time, and that is crucial for the fixed point method we use. In this problem any t_0 will work and we simply let $t_0 = 0$. Here are the details.

LEMMA 5. Let $\psi : [-h, 0] \to R$ be a given continuous initial function and let $z(t)$ be the unique solution determined by ψ. Then z exists on $[0, \infty)$.

Proof. The conditions of the example ensure that there is a unique local solution. It is well known that the only way in which a solution can fail to be defined for all future time is for there to exist $T > 0$ with $\limsup_{t \to T^-} |z(t)| = +\infty$. In particular, then $z(t-h)$ is a bounded continuous function on $[0, T]$. Now $a(t) > 0$ so there is $A > 0$ with $a(t) > A$ on $[0, T]$ and there is a $B > 0$ with $B > |b(t)||z(t-h)|$ on that same interval. If we define

$$
V(t, x) = (1 + |x|)e^{-Kt}
$$

then the derivative of V along any solution $x(t)$ of (11.1) satisfies

$$
\begin{aligned}
V' &\leq e^{-Kt}[-a(t)|x|^3 + |b(t)x(t-h)||x|^2 - K - K|x|] \\
&\leq e^{-Kt}[-A|x|^3 + Bx^2 - K] \leq 0
\end{aligned}
$$

for large enough K. Thus, $V(t, x(t))$ is bounded on $[0, T]$ and so it can not happen that $\limsup_{t \to T^-} |x(t)| = \infty$.

There will, of course, be problems in which we will be unable to find such a Liapunov function. In such cases we must state that solutions defined for all future time tend to zero.

We are not going to do the work of Lemma 4 here. Instead, for brevity we ask that

$$|b(t)| \leq ka(t), k < 1. \tag{11.2}$$

THEOREM 11.1. Let (11.2) hold and let $a(t) \geq c > 0$. For a given initial function $\eta : [-h, 0] \to R$, either:

a) $x^2(t, 0, \eta) \in L^1[0, \infty)$ or

b) $x(t, 0, \eta) \to 0$ as $t \to \infty$.

Proof. We let $z(t)$ be the unique solution of (11.1) and consider

$$x' = -a(t)z^2(t)x + b(t)z^2(t)x(t-h) \tag{11.3}$$

so that the solution can be written as

$$x(t) = \eta(0)e^{-\int_0^t a(s)z^2(s)ds} + \int_0^t e^{-\int_s^t a(u)z^2(u)du} b(s)z^2(s)x(s-h)ds.$$

Then we define our set S by

$$S = \{\phi : [-h, \infty) \to R | \phi(t) = \eta(t) \text{ on } [-h, 0], \phi(t) \to 0 \text{ as } t \to \infty, \phi \in C\}$$

and the mapping P (which will map $S \to S$ if a) fails) by $(P\phi)(t) = \eta(t)$ if $-h \leq t \leq 0$ and

$$(P\phi)(t) = \eta(0)e^{-\int_0^t a(s)z^2(s)ds} + \int_0^t e^{-\int_s^t a(u)z^2(u)du} b(s)z^2(s)ds\phi(s-h)ds$$

for $t \geq 0$. Our condition (11.2) readily yields that this is a contraction. Now, if z^2 is not in $L^1[0, \infty)$, then $\int_0^\infty a(t)z^2(t)dt = \infty$ and we can verify that $P : S \to S$. Hence, there is a unique fixed point in S, which is the unique solution z, and it goes to zero. This completes the proof.

REMARK 11.1. For completeness of our discussion of (E), consider

$$x'(t) = -a(t)x^3(t) + b(t)x^2(t-h)x. \tag{11.4}$$

We assume that

$$a(t) \geq c > 0 \tag{11.5}$$

and that

$$a(t-h) \geq k|b(t)|, k > 1. \tag{11.6}$$

From the argument in the previous section one readily concludes that for each continuous initial function, a unique solution exists on $[0, \infty)$. The next result is proved directly from the variation of parameters formula.

THEOREM 11.2. Let (11.5) and (11.6) hold. For a given continuous $\eta : [-h, 0] \to R$ either:

a) $x^2(t, 0, \eta) \in L^1[0, \infty)$ or

b) $x(t, 0, \eta) \to 0$ as $t \to \infty$.

12. EQUATION (E), ITSELF

We return to (E) which we write as

$$x' = -a(t)x^3(t) + b(t)x^3(t-h) \tag{12.1}$$

with a view to showing that solutions tend to 0 without requiring that $a(t)$ be bounded. For brevity we ask that

$$|b(t+h)| \le \gamma a(t), a(t) > 0, \gamma < 1. \tag{12.2}$$

(With more work we could ask that there is a t_0 with $\int_{t_0}^t [-a(s) + |b(s+h)|]ds \le 0$.) Under this assumption we can use the Liapunov function

$$V(t, x_t) = |x| + \int_{t-h}^t |b(s+h)||x^3(s)|ds$$

and obtain

$$V' \le [-a(t) + |b(t+h)|]|x^3|$$

which yields stability of the zero solution. This means that if $z(t)$ is the unique solution of (12.1) with continuous initial function $\eta : [-h, 0] \to R$, we can make z as small as we please by taking $|\eta|$ small. Thus, we ask that there exists $M > 0$ with

$$\int_{t-h}^t |b(u+h)|du \le M, \tag{12.3}$$

then find $\beta > 0$ with

$$2\beta + \gamma =: \alpha < 1, \tag{12.4}$$

and finally take η so small that

$$\int_{t-h}^t |b(u+h)|x^2(u, 0, \eta)du \le \beta. \tag{12.5}$$

While this does place an integral bound on b, there is no bound on a.

THEOREM 12.1. Let (12.2) to (12.5) hold with η selected and let $a(t) \geq c > 0$. Either:

a) $x^2(t, 0, \eta) \in L^1[0, \infty)$ or

b) $x(t, 0, \eta) \to 0$ as $t \to \infty$.

Proof. Let $z(t) = x(t, 0, \eta)$ which is the unique solution of

$$x' = -a(t)z^2(t)x + b(t)z^2(t-h)x(t-h), x_0 = \eta. \tag{12.6}$$

Define

$$S = \{\phi : [-h, \infty) \to R | \phi(t) = \eta(t) \text{ on } [-h, 0], \phi \in C, \phi(t) \to 0 \text{ as } t \to \infty\}$$

and define $(P\phi)(t) = \eta(t)$ for $-h \leq t \leq 0$ and for $t \geq 0$ define

$$(P\phi)(t) = \eta(0)e^{-\int_0^t a(s)z^2(s)ds}$$

$$+ \int_0^t e^{-\int_s^t a(u)z^2(u)du} b(s)z^2(s-h)\phi(s-h)ds.$$

If a) fails, then $\int_0^t a(s)z^2(s)ds \to \infty$ and we can show that $P : S \to S$. Now it seems to require a trick from Liapunov theory to show that we have a contraction. Note that

$$b(s)z^2(s-h)\phi(s-h)$$
$$= b(s+h)z^2(s)\phi(s) - b(s+h)z^2(s)\phi(s) + b(s)z^2(s-h)\phi(s-h)$$
$$= b(s+h)z^2(s)\phi(s) - (d/ds)\int_{s-h}^s b(u+h)z^2(u)\phi(u)du.$$

Hence,

$$(P\phi)(t) = \eta(0)e^{-\int_0^t a(s)z^2(s)ds}$$
$$+ \int_0^t e^{-\int_s^t a(u)z^2(u)du} b(s+h)z^2(s)\phi(s)ds$$
$$- \int_0^t e^{-\int_s^t a(u)z^2(u)du}(d/ds)\int_{s-h}^s b(u+h)z^2(u)\phi(u)du.$$

We integrate that last term by parts to obtain

$$-e^{-\int_s^t a(u)z^2(u)du}\int_{s-h}^s b(u+h)z^2(u)\phi(u)du\Big|_0^t$$

$$+ \int_0^t e^{-\int_s^t a(u)z^2(u)du} a(s)z^2(s)\int_{s-h}^s b(u+h)z^2(u)\phi(u)duds$$

$$= -\int_{t-h}^{t} b(u+h)z^2(u)\phi(u)du + e^{-\int_0^t a(u)z^2(u)du} \int_{-h}^{0} b(u+h)z^2(u)\eta(u)du$$

$$+ \int_0^t e^{-\int_s^t a(u)z^2(u)du} a(s)z^2(s) \int_{s-h}^{s} b(u+h)z^2(u)\phi(u)duds.$$

Now, it follows readily that we have a contraction.

13. TWO QUICK EXAMPLES

We want to cover the obvious perturbations of (E) so we look at two of those which are very simple.

First, let a and b be continuous and consider

$$x'(t) = -a(t)x^2(t-h)x + b(t)x^3(t-h). \tag{13.1}$$

It is readily argued that solutions can be defined for all future time, regardless of the signs of the functions. Let $\eta : [-h, 0] \to R$ be a given continuous initial function and let $z(t) = x(t, 0, \eta)$ which is the unique solution of

$$x' = -a(t)z^2(t-h)x + b(t)z^2(t-h)x(t-h), x_0 = \eta. \tag{13.2}$$

Define S as in earlier sections and define

$$(P\phi)(t) = \eta(0)e^{-\int_0^t a(s)z^2(s-h)ds} + \int_0^t e^{-\int_s^t a(u)z^2(u-h)du} b(s)z^2(s-h)\phi(s-h)ds.$$

We shall ask that

$$|b(t)| \leq k|a(t)|, k < 1. \tag{13.3}$$

THEOREM 13.1. If (13.3) holds and if $a(t) \geq c > 0$, either $x^2(t, 0, \eta) \in L^1[0, \infty)$ or $x(t, 0, \eta) \to 0$ as $t \to \infty$.

Finally, let $a(t) > 0$ be a continuous function and consider

$$x' = -a(t)x(t-h)x^2(t).$$

If $a(t) \geq c > 0$, then solutions can have finite escape time. Thus, let $\eta : [-h, 0] \to (0, \infty)$ and let $z(t) = x(t, 0, \eta)$. Notice that

$$x' = -[a(t)z(t-h)z(t)]x(t)$$

is linear with negative coefficient so the solution decreases monotonically. It is never zero by the uniqueness theorem. We have

$$x(t) = \eta(0)e^{-\int_0^t a(s)z(s-h)z(s)ds},$$

and, unless $z(s-h)z(s) \in L^1$, then $x(t) \to 0$. Indeed, $x(t)$ always tends to zero if $\eta > 0$.

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