Fixed Point Theory, 10(2009), No.2, ...-...
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

FIXED POINTS AND FRACTIONAL DIFFERENTIAL EQUATIONS: EXAMPLES

T. A. BURTON[1] AND BO ZHANG[2]

[1]Northwest Research Institute, 732 Caroline St., Port Angeles 98362, WA USA
[2]Department of Mathematics and Computer Science, Fayetteville State University, Fayetteville, NC 28301
E-mail: taburton@olypen.com, bzhang@uncfsu.edu

Abstract. We study a fractional differential equation of Caputo type by first inverting it as an integral equation, then noting that the kernel is completely monotone, and finally transforming it into another integral equation with a kernel which supports both contractions and compact maps. That kernel allows us to use fixed point theory to obtain qualitative properties of solutions. At the end of Section 4 we give a list of five transformations which convert challenging problems into simple fixed point problems. We treat linear, superlinear, and sublinear examples using Krasnoselskii's fixed point theorem.

Key Words and Phrases: fractional differential equations, integral equations, fixed points
2000 Mathematics Subject Classification: Primary: 34A08, 47G05, 34D20

1. **Motivation and Introduction**

Krasnoselskii studied a paper by Schauder on partial differential equations and developed a working hypothesis: The inversion of a perturbed differential operator yields the sum of a contraction and compact map [10, p. 31]. As we all know, sometimes it does and sometimes it does not, but the idea has had a profound impact on both analysis and fixed point theory. Typical elementary functions which generate such maps are linear ($a(t)x$), superlinear (x^3), and sublinear ($x^{1/3}$), together with their obvious generalizations. As we strive to communicate ideas in this brief paper, we will focus on just such elementary examples. The inversion of a fractional differential equation of Caputo type with continuous functions is, in fact, nothing but a well-known integral equation with a kernel of the form $(t-s)^{q-1}$ which we readily recognize to be from a heat equation when $q = 1/2$. Thus, we are back to the Schauder problem of a partial differential equation and our interest in the sum of a contraction and a compact map is then well-motivated. The initial integral equation is totally unmanageable as either a contraction or a compact map. All would fail were it not for the fact that the kernel is completely monotone which makes it possible to trade that big kernel for $R(t-s)$ with the property that $0 < R(t)$ and $\int_0^\infty R(s)ds = 1$. Moreover, the new integral equations generate equi-continuous sets. From that good fortune, we now have a kernel which strongly promotes both contractions and compact maps. A series of trades is listed at the end of Section 4. In every way, the problem has fallen into the exact category envisioned by Krasnoselskii in his study of Schauder's paper.

We consider a fractional differential equation of Caputo type

$$ {}^cD^q x = -u(t, x(t)), \quad 0 < q < 1, \tag{1} $$

with a view to establishing qualitative properties of solutions by means of fixed point theory. When $u(t, x)$ is continuous then (1) is immediately inverted as the very familiar integral equation ([7, p. 54], [6, pp. 78, 86, 103])

$$ x(t) = x(0) - \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} u(s, x(s)) ds \tag{2} $$

where Γ is the gamma function.

The kernel is singular and is not in $L^1[0, \infty)$ and this leads to both difficulties and great simplifications. For example, if $u(t, x)$ has a term $f : [0, \infty) \to \Re$, then

$$ F(t) := \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s) ds $$

can be unbounded when f is bounded. Our topic here concerns fixed point methods. Virtually never can (2) be used to define a mapping which is either a contraction or a compact map because of those properties of the kernel.

On the other hand, if we contrive to write $u(t,x) = x + G(t,x)$, then a long line of beautiful properties emerge. So much so that the investigator is tempted to write $u(t,x) = x-(x-u(t,x))$ regardless of how unsuited the union $x - u(t,x)$ may prove to be in later investigation. The reason is simple and we discussed this with full details in [5]. Here is a sketch. If $u(t,x) = x + G(t,x)$ then we have

$$x(t) = x(0) - \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1}[x(s) + G(s,x(s))]ds \tag{3}$$

and we can follow Miller [8, pp. 191-192] and decompose this as the linear part

$$z(t) = x(0) - \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} z(s)ds \tag{4}$$

and

$$x(t) = z(t) - \int_0^t R(t-s)G(s,x(s))ds. \tag{5}$$

Here, $R(t)$ is the resolvent solving

$$R(t) = \frac{t^{q-1}}{\Gamma(q)} - \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} R(s)ds. \tag{6}$$

There is then the variation of parameters formula yielding

$$z(t) = x(0) - \int_0^t R(t-s)x(0)ds. \tag{7}$$

The fact that the kernel is completely monotone and not in $L^1[0,\infty)$ allows us to follow Miller [8, pp. 212-3, 221-4] and deduce that

$$0 \le R(t) \le \frac{1}{\Gamma(q)} t^{q-1}, \quad \int_0^\infty R(s)ds = 1, \tag{8}$$

and that $R(t)$ is also completely monotone. For reference here, a function, say $R(t)$, is completely monotone if $(-1)^k R^{(k)}(t) \ge 0$ for $k = 0, 1, 2, ...$ and $0 < t < \infty$.

Now, when we focus on (5) with (8) holding we readily see that all of our aforementioned problems with (2) have vanished. There now emerge reasons that fixed point theory is a natural tool for investigation of this equation.

The large kernel in (2) quite effectively prevents (2) from defining either a contraction or a compact map. But (5) supports both.

First, if we want to show that solutions tend to zero, then the $x(0)$ in (3) is a problem, whereas in (5) we have traded $x(0)$ for $z(t)$ and from (7) we see that $z(t) \to 0$.

Next, suppose that $G(t,x) = G_1(t,x) + G_2(t,x)$ where $G_1(t,x)$ can define a contraction mapping while $G_2(t,x)$ is bounded over some set of functions.

(i) We see that $z(t) - \int_0^t R(t-s)G_1(s,x(s))ds$ still defines that contraction on the same set of functions.

(ii) Moreover, we will prove that $\int_0^t R(t-s)G_2(s,x(s))ds$ will map that set of functions into an equicontinuous set.

Finally, $G(t,x)$ itself may contain a large function, say $a(t)$, which prevents it from defining a contraction. Because t^{q-1} is completely monotone, it is possible to absorb the average value, say J, of that function $a(t)$ harmlessly into the kernel with the result that $a(t)$ is replaced by a value smaller than one (a contraction constant) and all the other terms in G are also divided by J.

These are the properties which we will formally demonstrate in the following pages. The overall idea of the paper is to develop contractions and compact maps which we finally combine with Krasnoselskii's fixed point theorem.

2. The linear equation

We begin with the scalar equation

$$ {}^cD^q x = f(t) - a(t)x(t), \quad 0 < q < 1, \quad x(0) = x_0, \tag{9} $$

inverted as

$$ x(t) = x(0) - \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1}[a(s)x(s) - f(s)]ds \tag{10} $$

where $a, f : [0,\infty) \to \Re$ are continuous and there are positive numbers ϵ and M with

$$ 0 < \epsilon \le a(t) \le M. \tag{11} $$

We emphasize that (10) would fail to define a contraction on the space of bounded continuous functions because of the large kernel and because $a(t)$ is allowed to be arbitrarily large. We will exchange $R(t-s)$ for the kernel, but first we will reduce $a(t)$ to a function bounded by $\alpha < 1$. The proof of this simple, but fundamental, result will provide the basis for much of the sequel.

Here, BC denotes the Banach space of bounded continuous functions $\psi : [0,\infty) \to \Re$ with the supremum norm, $\|\cdot\|$.

Theorem 2.1. *Let (11) hold. If $f \in BC$ then for every $x(0) \in \Re$ there is a unique solution $x(t)$ of (10) and it is also in BC. If $f \in L^1[0,\infty)$ or if $f(t) \to 0$ as $t \to \infty$, then $x(t) \to 0$.*

Proof. Define $J = \epsilon + (1/2)(M - \epsilon)$. Then there is an α with

$$J > 0, \quad 0 < \alpha < 1, \quad |a(t) - J| < \alpha J. \tag{12}$$

We have

$$\begin{aligned} x(t) &= x(0) - \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1}[Jx(s) + (a(s) - J)x(s) - f(s)]ds \\ &= x(0) - \frac{1}{\Gamma(q)} \int_0^t J(t-s)^{q-1}[x(s) + \frac{(a(s) - J)}{J}x(s) - \frac{f(s)}{J}]ds \end{aligned}$$

which we decompose into

$$z(t) = x(0) - \frac{J}{\Gamma(q)} \int_0^t (t-s)^{q-1} z(s)ds$$

with solution

$$z(t) = x(0) - \int_0^t R(t-s)x(0)ds$$

and

$$x(t) = z(t) + \frac{1}{J}\int_0^t R(t-s)f(s)ds - \int_0^t R(t-s)\frac{(a(s) - J)}{J}x(s)ds.$$

This will define our contraction mapping on BC. For $\psi \in BC$ we define $P : BC \to BC$ by

$$(P\psi)(t) = z(t) + \frac{1}{J}\int_0^t R(t-s)f(s)ds - \int_0^t R(t-s)\frac{(a(s) - J)}{J}\psi(s)ds.$$

It is clear that the mapping is into BC. Moreover, it is a contraction because of (12). Thus, the first conclusion holds. The second conclusion is obtained by adding to BC the condition that the $\psi \to 0$. $\square$

3. A one-step problem

We now consider

$$^{c}D^{q}x = -\sin x(t) + bx(t-r), \quad 0 < q < 1, \quad r > 0, \quad x_0 = \phi \tag{12}$$

where $\phi : [-r, 0] \to \Re$ is a continuous initial function so that $x_0 = \phi$ means that $x(t) = \phi(t)$ for $-r \le t \le 0$. We seek a solution with $x(0) = \phi(0)$. It is a non-trivial example of a problem mentioned in the introduction. As the reader works through the proof it becomes clear that this is one of a great class of problems which can be solved in this way. The inverted equation is

$$x(t) = \phi(0) - \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}[\sin x(s) - bx(s-r)]ds$$

in which there is no chance of a contraction. We simply add and subtract $x(s)$ to the integrand, decompose into two equations, and we immediately have a straightforward contraction without any modifications of the original contraction mapping theorem.

Theorem 3.1. *If $|b|\pi/4 + \|\phi\| < \sqrt{2}/2$ and $|b| < \sqrt{2}/2$, then the solution satisfies $|x(t,\phi)| < \pi/4$ for $t > 0$.*

Proof. Let $(S, \|\cdot\|)$ be the complete metric space with

$$S = \{\psi : [-r, \infty) \to [-\pi/4, \pi/4] | \psi(t) = \phi(t), -r \le t \le 0\}$$

with the supremum metric. Also, $\|\phi\|$ denotes the supremum of ϕ on $[-r, 0]$. We have

$$\begin{aligned} x(t) &= \phi(0) - \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}[\sin x(s) - bx(s-r)]ds \\ &= \phi(0) - \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}[x(s) - (x(s) - \sin x(s)) - bx(s-r)]ds. \end{aligned}$$

Separate the equation as

$$z(t) = \phi(0) - \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}z(s)ds,$$

$$z(t) = \phi(0) - \int_0^t R(t-s)\phi(0)ds, \quad \|z\| \le \|\phi\|,$$

$$x(t) = z(t) + \int_0^t R(t-s)[x(s) - \sin x(s) + bx(s-r)]ds.$$

Notice that since $x - \sin x$ is increasing on $[-\pi/4, \pi/4]$, if $\|\psi\| \leq \pi/4$ then

$$|\psi(t) - \sin \psi(t)| \leq \pi/4 - \sqrt{2}/2$$

so

$$|\psi(s) - \sin \psi(s) + b\psi(s-r)| \leq \pi/4 - \sqrt{2}/2 + |b|\pi/4.$$

Define $P : S \to S$ by $\psi \in S$ implies that $(P\psi)(t) = \phi(t)$ for $-r \leq t \leq 0$ and

$$(P\psi)(t) = z(t) + \int_0^t R(t-s)[\psi(s) - \sin \psi(s) + b\psi(s-r)]ds$$

for $t \geq 0$ with

$$\begin{aligned}|(P\psi)(t)| &\leq \|\phi\| + (1+|b|)\pi/4 - \sqrt{2}/2 \\ &< \pi/4 + \sqrt{2}/2 - \sqrt{2}/2 = \pi/4\end{aligned}$$

so $P\psi \in S$.

A contraction constant for $x - \sin x$ is $1 - \sqrt{2}/2$, while a contraction constant for $b\psi(t-r)$ is $|b|$. Thus, we need

$$1 - \sqrt{2}/2 + |b| < 1$$

which holds by assumption. Hence, the mapping has a unique fixed point which solves the problem. □

4. Large contractions

A typical superlinear term is x^3. That function defines a very nice contraction for small x, but we need to write it as $x - (x - x^3)$ so that we will have the x to bring us to the R integral. However, that leaves us with $x - x^3$ and it loses its contraction property as $x \to 0$. It turns out that this does not matter. That function is still a **large contraction** [1] and it will have a unique fixed point.

The following definition and theorem are needed for our next result. They are found in [1].

Definition 4.1. Let (M, ρ) be a metric space and $B : M \to M$. B is said to be a *large contraction* if φ, $\psi \in M$, with $\varphi \neq \psi$ then $\rho(B\varphi, B\psi) < \rho(\varphi, \psi)$ and if $\forall \varepsilon > 0$ $\exists \delta < 1$ such that $[\varphi, \psi \in M,\ \rho(\varphi, \psi) \geq \varepsilon] \Rightarrow \rho(B\varphi, B\psi) \leq \delta\rho(\varphi, \psi)$.

Theorem 4.2. *Let (M, ρ) be a complete metric space and B be a large contraction. Suppose there is an $x \in M$ and an $L > 0$, such that $\rho(x, B^n x) \leq L$ for all $n \geq 1$. Then B has a unique fixed point in M.*

Theorem 4.3. *If $|x(0)| < \sqrt{3}/9$ then the solution of*

$$^cD^q x = -x^3 \tag{13}$$

is bounded.

Proof. Invert the equation as

$$x(t) = x(0) - \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1}[x(s) - (x(s) - x^3(s))]ds$$

and separate it into

$$z(t) = x(0) - \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} z(s)ds$$

(so that $|z(t)| \leq |x(0)| < \sqrt{3}/9$) and

$$x(t) = z(t) + \int_0^t R(t-s)[x(s) - x^3(s)]ds.$$

A maximum of $y = x - x^3$ of $2\sqrt{3}/9$ occurs at $x = 1/\sqrt{3}$. If we define

$$S = \{\phi : [0, \infty) \to \Re | \|\phi\| \leq \sqrt{3}/3, \phi \, \text{continuous}\} \tag{14}$$

and P by $\phi \in S$ implies

$$(P\phi)(t) = z(t) + \int_0^t R(t-s)[\phi(s) - \phi^3(s)]ds \tag{15}$$

then $|(P\phi)(t)| \leq |x(0)| + (2\sqrt{3}/9) \leq \sqrt{3}/3$ so $P\phi \in S$.

To see that P defines a large contraction in S, consider $y = x - x^3$ with derivative $y' = 1 - 3x^2$ which will give us the contraction constant at any value of x. It is not difficult to see (and it is discussed in [1] with more detail in [4, p. 23]) that this is a large contraction and that the mapping has a unique fixed point. □

Theorem 4.4. *The zero solution of (13) is stable.*

Proof. We invert (13) as

$$x(t) = x(0) - \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1}[x(s) - (x(s) - x^3(s))]ds$$

which we write as

$$x(t) = x(0) - \int_0^t C(t-s)[x(s) - (x(s) - x^3(s))]ds$$

with

$$C(t) = \frac{1}{\Gamma(q)} t^{q-1} \quad \text{for} \quad t > 0. \tag{16}$$

We separate it into

$$z(t) = x(0) - \int_0^t C(t-s)z(s)ds$$

with solution

$$z(t) = x(0)[1 - \int_0^t R(s)ds] \tag{17}$$

and

$$x(t) = z(t) + \int_0^t R(t-s)[x(s) - x^3(s)]ds \tag{18}$$

where again the resolvent $R(t)$ satisfies

$$R(t) = C(t) - \int_0^t C(t-s)R(s)ds \tag{19}$$

having property (8).

Let $0 < \epsilon \leq \sqrt{3}/3$ and choose $\delta > 0$ such that $\delta < \epsilon^3$. Let $x(t)$ be a solution of (13) with $|x(0)| < \delta$. We claim that $|x(t)| < \epsilon$ for all $t \geq 0$. In fact, if there exists a $\bar{t} > 0$ such that $|x(\bar{t})| = \epsilon$ with $|x(s)| < \epsilon$ for $0 \leq s < \bar{t}$, we obtain from (18) that

$$\begin{aligned}
|x(\bar{t})| &\leq |z(\bar{t})| + \int_0^{\bar{t}} R(\bar{t}-s) \left| x(s) - x^3(s) \right| ds \\
&= |z(\bar{t})| + \int_0^{\bar{t}} R(\bar{t}-s)[|x(s)| - |x^3(s)|]ds \\
&\leq |x(0)| + (\epsilon - \epsilon^3) \int_0^{\bar{t}} R(\bar{t}-s)ds \\
&\leq \delta + \epsilon - \epsilon^3 < \epsilon
\end{aligned}$$

which contradicts the definition of $x(\bar{t})$. Here we have used the fact that $y = r - r^3$ is increasing on $[0, \sqrt{3}/3]$ with a maximum value of $2\sqrt{3}/9$. Thus, the zero solution of (13) is stable. $\square$

Theorem 4.5. *The zero solution of (13) is asymptotically stable.*

Proof. It is shown in Theorem 4.4 that the zero solution of (13) is stable. Thus, we need to show here that $x(t) \to 0$ as $t \to \infty$ for any solution $x(t)$ of (13) with $|x(0)| + (2\sqrt{3}/9) \leq \sqrt{3}/3$. Now for S given in (14), we define

$$S_0 = \{\phi \in S \mid \phi(t) \to 0 \text{ as } t \to \infty\}.$$

Then S_0 is a complete metric space with the supremum metric $\rho(x, y) = \|x - y\|$. Let the mapping P be defined in (15). Then for $\phi \in S_0$, we have $P\phi \in S$. We observe that $\int_0^\infty R(s)ds = 1$ implies that

$$z(t) = x(0)[1 - \int_0^t R(s)ds] = x(0) \int_t^\infty R(s)ds \to 0 \text{ as } t \to \infty.$$

Also, since $R \in L^1(0, \infty)$ and $\phi(t) \to 0$ as $t \to \infty$, we have

$$\int_0^t R(t-s)[\phi(s) - \phi^3(s)]ds \to 0 \text{ as } t \to \infty$$

and, hence, $(P\phi)(t) \to 0$ as $t \to \infty$. This yields that $P\phi \in S_0$. By the proof of Theorem 4.3, we also see that P is a large contraction on S_0 and $\|P^n\phi\| \leq \sqrt{3}/3$ for all $\phi \in S_0$ and $n \geq 1$. By Theorem 4.2, P has a unique fixed point $\phi \in S_0$ which is a solution of (13) with $\phi(0) = x(0)$ and $\phi(t) \to 0$ as $t \to \infty$. Thus, the zero solution of (13) is asymptotically stable. $\square$

Remark One may note that this is a general result. Everything would work for

$${}^cD^q x = -x^{2n+1}, 0 < q < 1.$$

In fact, we can consider

$${}^cD^q x = -g(x), 0 < q < 1,$$

where $g(0) = 0$. We define $G(x) = x - g(x)$ and ask that $dG(x)/dx$ is continuous and positive except, possibly, at $x = 0$ on an interval $[-b, b]$, and $|dG(x)/dx| \leq \alpha < 1$.

The work with the linear equation ${}^cD^q = f(t) - a(t)x(t)$ with $a(t)$ large was not restricted to linear equations. It works perfectly on

$${}^cD^q x = -a(t)x^3, \quad 0 < q < 1,$$

with $0 < \epsilon \le a(t) < M$. In fact, a very interesting addition occurs. We obtain a large contraction plus an ordinary contraction with contraction constant $\alpha < 1$ and it is precisely this property which allows us to add the two contractions together and obtain a large contraction. Here are the details.

Proof. Invert the equation as

$$x(t) = x(0) - \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} a(s) x^3(s) ds.$$

Exactly as in Section 2, we find $J > 0, \alpha < 1$ with $|J - a(t)| \le \alpha J$. Then

$$\begin{aligned} a(s)x^3(s) &= Jx^3(s) - (Jx^3(s) - a(s)x^3(s)) \\ &= J[x^3(s) - \frac{(J-a(s))}{J} x^3(s)] \\ &= J[x(s) - (x(s) - x^3(s)) - \frac{(J-a(s))}{J} x^3(s)]ds \end{aligned}$$

so

$$\begin{aligned} x(t) = x(0) - \frac{J}{\Gamma(q)} \int_0^t (t-s)^{q-1} &[x(s) - (x(s) - x^3(s)) \\ &- \frac{(J-a(s))}{J} x^3(s)]ds \end{aligned}$$

which we decompose as

$$z(t) = x(0) - \frac{J}{\Gamma(q)} \int_0^t (t-s)^{q-1} z(s) ds$$

and

$$x(t) = z(t) + \int_0^t R(t-s)[(x(s) - x^3(s)) + \frac{(J-a(s))}{J} x^3(s)]ds.$$

That integrand will be a large contraction. The contraction constant for the first term is obtained from $y = x - x^3$ with $y' = 1 - 3x^2$, while the x derivative of $\frac{(J-a(s))}{J} x^3$ is bounded by $3\alpha x^2$, yielding the sum $1 - 3x^2 + 3\alpha x^2$. This will yield a large contraction on a certain space of bounded continuous functions. □

Summary

There are a number of trades happening and it is worth listing them here. Suppose, for example, that we begin with

$$^cD^q x = -a(t)x^3$$

with $a(t)$ as above, and invert it as

$$x(t) = x(0) - \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} a(s)x^3(s)ds.$$

1. The local contraction properties of x^3 are destroyed by a large $a(t)$, so we exchange $a(t)$ for a function bounded by a constant smaller than 1.

2. The contraction is still destroyed by the large kernel so we must add and subtract $x(s)$ in order to get a linear part so that we can exchange the large kernel for $R(t-s)$.

3. But in the process of the last item, the contraction function x^3 is replaced by $x - x^3$, so we exchange the idea of a contraction for the idea of a large contraction.

4. Now, we exchange the large kernel $(t-s)^{q-1}$ for the small kernel $R(t-s)$.

5. When we want to prove that solutions tend to zero by fixed point methods, then $x(0)$ can be a problem. But we trade $x(0)$ for $z(t)$ which tends to zero as $t \to \infty$.

With this series of exchanges we now have a simple fixed point problem.

5. Compact maps: Krasnoselskii's theorem

We have focused on

$$z(t) = x(0) - \int_0^t R(t-s)x(0)ds$$

and

$$x(t) = z(t) - \int_0^t R(t-s)G(s, x(s))ds$$

where G satisfies some type of contraction condition. With a generalization of Krasnoselskii's theorem [1] in mind we now investigate the possibility of treating

$$x(t) = z(t) - \int_0^t R(t-s)[G(s, x(s)) + g(s, x(s))]ds$$

where G defines a large contraction and

$$\int_0^t R(t-s)g(s,x(s))ds$$

defines a compact map. Thus, we are interested in conditions under which an integral will be equi-continuous.

Theorem 5.1. *There is a constant $H > 0$ so that if $0 < t_2 < t_1 < \infty$, if $x \in BC$ with $|g(t,x(t))| \leq K$, and if $0 < q < 1$ then*

$$\begin{aligned} L :&= \left| \int_0^{t_1} R(t_1-s)g(s,x(s))ds - \int_0^{t_2} R(t_2-s)g(s,x(s))ds \right| \\ &\leq H|t_1-t_2|^q. \end{aligned}$$

Proof. Note that since $R(t)$ is decreasing and there is a constant D with $0 \leq R(t) \leq Dt^{q-1}$ we have

$$\begin{aligned} L &\leq \int_0^{t_2} |R(t_1-s)-R(t_2-s)||g(s,x(s))|ds + \int_{t_2}^{t_1} |R(t_1-s)||g(s,x(s))|ds \\ &\leq \int_0^{t_2} K[R(t_2-s)-R(t_1-s)]ds + K\int_{t_2}^{t_1} R(t_1-s)ds \\ &= K\int_0^{t_2} R(t_2-s)ds - K\int_0^{t_2} R(t_1-s)ds + K\int_{t_2}^{t_1} R(t_1-s)ds \\ &= K\int_0^{t_2} R(s)ds - K\int_{t_1-t_2}^{t_1} R(s)ds + K\int_{t_2}^{t_1} R(t_1-s)ds \\ &= K\int_0^{t_2} R(s)ds - K\int_0^{t_1} R(s)ds + 2K\int_{t_2}^{t_1} R(t_1-s)ds \end{aligned}$$

the sum of the first two terms is negative

$$\begin{aligned} &\leq 2DK\int_{t_2}^{t_1} (t_1-s)^{q-1}ds \\ &= -2(KD/q)(t_1-s)^q\Big|_{t_2}^{t_1} \\ &= (2KD/q)(t_1-t_2)^q. \end{aligned}$$

□

Krasnoselskii combined the contraction mapping theorem with Schauder's theorem. We extended it in [1], replacing contraction by large contraction, as follows. See also [2] for different conditions in (i). Krasnoselskii's result differs in asking that $A, B : M \to S$.

Theorem 5.2. *Let $(S, \|\cdot\|)$ be a Banach space, M a closed, bounded, convex, nonempty subset of S. Suppose that $A,\ B : M \to M$ and that*

$$x, y \in M \Rightarrow Ax + By \in M, \tag{i}$$

$$A \text{ is continuous and } AM \text{ is contained in a compact subset of } M, \tag{ii}$$

$$B \text{ is a large contraction.} \tag{iii}$$

Then $\exists\ y \in M$ with $Ay + By = y$.

We will give an example now using only the contraction instead of large contraction so that there will not be so many details. We could use the term $a(t)x^3$ instead of $a(t)x$ and that would lead us to the large contraction, but with so many more details. The reason for this theorem is that $\phi(t)x^{1/3}$ is not Lipschitz so the contractions fail.

Investigators have found it difficult to verify (i) in many problems [2]. The reader may notice that for contractions and sublinear terms, there is an automatic way to show (i).

Theorem 5.3. *Let $a, \phi : [0, \infty) \to \Re$ be continuous, $\phi(t) \to 0$ as $t \to \infty$, $0 < \epsilon \le a(t) \le M$. For $J = (1/2)(M + \epsilon)$, find $\alpha < 1$ with $|J - a(t)| \le \alpha J$. There is a solution of*

$$^cD^q x(t) = -a(t)x(t) + \phi(t)x^{1/3}(t), \quad 0 < q < 1,\ x(0) \in \Re,$$

which tends to zero as $t \to \infty$.

Proof. We invert and decompose our equation just as before and have

$$x(t) = z(t) + \int_0^t R(t-s)\left[\frac{J - a(s)}{J}x(s) + \frac{\phi(s)}{J}x^{1/3}(s)ds\right].$$

Let $(X, \|\cdot\|)$ be the Banach space of bounded continuous functions $\psi : [0, \infty) \to \Re$ with the supremum norm. For an $H > 1$ to be determined, let S be the subset of X with $\|\psi\| \le H$ and $\psi(t) \to 0$ as $t \to \infty$. Define $P : S \to S$ by $\psi \in S$ implies that

$$(P\psi)(t) = z(t) + \int_0^t R(t-s)\left[\frac{J - a(s)}{J}\psi(s) + \frac{\phi(s)}{J}\psi^{1/3}(s)\right]ds.$$

Referring to Krasnoselskii's theorem we define

$$(Bx)(t) = z(t) + \int_0^t R(t-s)\frac{J-a(s)}{J}x(s)ds$$

and

$$(Ax)(t) = \int_0^t R(t-s)\frac{\phi(s)}{J}x^{1/3}(s)ds.$$

It follows from our previous theorem that $B : S \to S$ is a contraction mapping and that AS is equicontinuous. Since the functions

$$|(Ax)(t)| \leq \int_0^t R(t-s)\frac{|\phi(s)|}{J}H^{1/3}ds \to 0$$

as $t \to \infty$ for any $x \in M$, we see that AS is also in a compact subset of X.

Notice that B will shrink ψ in the sense that if $\psi \in S$ then

$$\|B\psi\| \leq |x(0)| + \alpha\|\psi\| \leq |x(0)| + \alpha H.$$

Next, notice that when $H > 1$ then A will shrink ψ in the sense that if $\|\psi\| \leq H$ then

$$|(A\psi)(t)| \leq \frac{\|\phi\|}{J}H^{1/3}\int_0^t R(t-s)ds \leq \frac{\|\phi\|}{J}H^{1/3}$$

and hence,

$$\|A\psi\| \leq \frac{\|\phi\|}{J}H^{1/3}.$$

We need the property (i) so for $\psi, \eta \in S$ we need

$$\|A\psi\| + \|B\eta\| \leq |x(0)| + \alpha H + \frac{\|\phi\|}{J}H^{1/3} \leq H$$

or

$$|x(0)| + \frac{\|\phi\|}{J}H^{1/3} \leq (1-\alpha)H.$$

Let $|x(0)|, \|\phi\|, J > 0$ be fixed. As H increases, this can be satisfied and that will determine S such that (i) will be satisfied. By Krasnoselskii's theorem there is a fixed point, ψ, a solution of our equation. As $P\psi \to 0$ and $P\psi = \psi$, the conclusion follows. $\square$

6. Extensions

It is possible to prove a more general fixed point theorem so that in Theorem 2.1 we can add to $a(t)$ a continuous function $b(t) \in L^2[0,\infty)$ and retain the boundedness conclusion

In Theorem 5.3 we needed $\phi(t) \to 0$ in order to have a compact map. In recent years there have been several generalizations of the Krasnoselskii result which use weaker conditions to achieve the compactness, not the least of which is weighted norms.

We have said nothing here about L^p solutions, but the fixed point methods will produce them as well.

Finally, there has been much recent interest in the possibility of periodic solutions generated by periodic forcing functions. The existence of such solutions is now in great doubt, although some controversy surrounds the subject. But asymptotically periodic solutions are to be expected from periodic perturbations and these are readily obtained with fixed point methods.

Acknowledgments

Research of the second author was supported in part by the Fayetteville State University Faculty Summer Research Grant-2011.

References

[1] T. A. Burton, Integral equations, implicit functions, and fixed points, *Proc. Amer. Math. Soc.* 124(1996) 2383-2390.

[2] T. A. Burton, A fixed-point theorem of Krasnoselskii, *Appl. Math. Lett.*11(1998) 85-88.

[3] T. A. Burton, *Liapunov Functionals for Integral Equations*, Trafford Publishing, Victoria, B. C., Canada, 2008.

[4] T. A. Burton, *Stability by Fixed Point Theory for Functional Differential Equations*, Dover, Mineola, New York, 2006.

[5] T. A. Burton, Fractional differential equations and Lyapunov functionals, *Nonlinear Anal.* 74(2011) 5648-5662.

[6] Kai Diethelm, *The Analysis of Fractional Differential Equations*, Springer, New York, 2004.

[7] V. Lakshmikantham , S. Leela, J. Vasundhara Devi, *Theory of Fractional Dynamic Systems*, Cambridge Scientific Publishers, Cottenham, Cambridge, 2009.

[8] R. K. Miller, *Nonlinear Integral Equations*, Benjamin, Menlo Park, CA, 1971.

[9] G. E. H. Reuter, Über eine Volterrasche Integralgleichungen mit totalmonotonem Kern, *Archive der Matematik*, 7(1956), 59-66.

[10] D. R. Smart, *Fixed Point Theorems*, Cambridge Univ. Press, Cambridge, 1980.