FRACTIONAL EQUATIONS AND GENERALIZATIONS OF SCHAEFER'S AND KRASNOSELSKII'S FIXED POINT THEOREMS

T. A. BURTON AND BO ZHANG

ABSTRACT. We study a fractional differential equation of Caputo type by first transforming it into an integral equation with an $L^1[0,\infty)$ kernel and then applying fixed point theory of Banach and Schauder type using a weighted norm to avoid stringent compactness conditions. It becomes clear that tedious construction of mapping sets and boundedness conditions can be avoided if we use fixed point theorems of Schaefer and Krasnoselskii type. The weighted norm then produces open sets so large that it is difficult to show that mappings are compact. This then leads us to generalize both Schaefer's and Krasnoselskii's fixed point theorems which yield simple and direct qualitative results for the fractional differential equations. The weight, g, yields compactness, but it does much more. The generalized fixed point theorems now yield growth properties of the solutions of the fractional differential equations.

1. Introduction

In the study of fractional differential equations we arrive at an integral equation of the form

$$x(t) = F(t) + \int_0^t R(t-s)[u(s,x(s)) + v(s,x(s))]ds$$

where $u(t,x)$ satisfies a contraction condition while

$$(Ax)(t) = \int_0^t R(t-s)v(s,x(s))ds$$

maps bounded sets into equicontinuous sets. We wish to set up a fixed point mapping of the Krasnoselskii type so we need A to map a closed, bounded, convex set into a compact set. That can be very difficult using the supremum norm unless we find that there is a fixed function $\phi : [0,\infty) \to \Re$, $\phi(t) \to 0$ as $t \to \infty$, and for $\psi \in M$ then $|(A\psi)(t)| \le \phi(t)$.

That is a severe condition. If we change to a weighted norm then it is simple to show that A maps M into a compact set. In order to use that weighted norm we must verify that all of our continuity and

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contraction properties hold using the weighted norm. This paper shows how to put all of this together and then apply it to a concrete example.

That method works well for local results, particularly when we are dealing with highly nonlinear terms, such as x^3, where one would expect to see solutions with finite escape time if we were to allow arbitrary initial conditions. But if our problems are mildly nonlinear, if we want to allow unbounded forcing functions, and if we want a global solution, then fixed point theorems of the Schaefer type immediately come to mind. There is a big problem with such theorems because they ask that bounded sets map into compact sets. The problem is that bounded sets in that norm can be absolutely enormous. The task of the investigator is to trim those sets so that compactness will be as easily proved as it was in the problems mentioned above.

A major part of this paper, then, is to extend both Schaefer's fixed point theorem (see [18] or [19, pp. 29]) and Krasnoselskii's fixed point theorem (see [12] or [19, pp. 31]) to cover problems of this type.

2. Preparation

We consider a fractional differential equation of Caputo type

$$(1)\qquad {}^cD^q x(t) = -a(t)x + v(t, x(t)) + f(t), \quad 0 < q < 1, \quad x(0) \in \Re$$

where $v : [0,\infty) \times \Re \to \Re$, a, $f : [0,\infty) \to \Re$ are all continuous. Under these conditions (1) can be inverted as

$$(2)\quad x(t) = x(0) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} [-a(s)x(s) + v(s, x(s)) + f(s)]ds.$$

For the inversion see [13, p. 54], [14], or [9, pp. 78, 86, 103], for example. Conditions on f will be stated later, but generally we expect $f \in L^p[0,\infty)$ or $f \in BC$ where $(BC, \|\cdot\|)$ will always denote the Banach space of bounded continuous functions with the supremum norm. It is assumed that there are positive constants ϵ and L with

$$(3)\qquad 0 < \epsilon \le a(t) \le L.$$

It is then possible to find $\alpha < 1$ and $J > 0$ with

$$(4)\qquad J = \epsilon + (1/2)(L - \epsilon), \quad |a(t) - J| \le \alpha J.$$

Having determined J we rewrite (2) as

(5)
$$x(t) = x(0) + \frac{J}{\Gamma(q)} \int_0^t (t-s)^{q-1} \left[-x(s) + x(s) - \frac{a(s)x(s)}{J} + \frac{v(s, x(s))}{J} + \frac{f(s)}{J} \right] ds.$$

We come now to a highly technical part of the paper. The following formulation was given with detailed references in [4] and it all comes from parts of [15, pp. 205-224, 189-192]. We define the completely monotone function

$$C(t) = \frac{Jt^{q-1}}{\Gamma(q)} \tag{6}$$

and write the linear part of (5) as

$$z(t) = x(0) - \int_0^t C(t-s)z(s)ds \tag{7}$$

having resolvent equation

$$R(t) = C(t) - \int_0^t C(t-s)R(s)ds \tag{8}$$

with solution satisfying

$$0 < R(t) \le C(t), \quad \int_0^\infty R(t)dt = 1, \tag{9}$$

and R is also completely monotone. Moreover, the variation of parameters formula yields

$$z(t) = x(0)\left[1 - \int_0^t R(s)ds\right]. \tag{10}$$

In case it is needed, we mention that earlier we showed in [7] that for $x(0)$ nonzero then

$$z \in L^p \iff p > 1/q. \tag{11}$$

We can now use the nonlinear variation of parameters formula (see [15, pp. 189-192]) to write (5) as

$$x(t) = z(t) + \int_0^t R(t-s)\left[\frac{J-a(s))}{J}x(s) + \frac{v(s,x(s))}{J}\right]ds + F(t) \tag{12}$$

where

$$F(t) = \frac{1}{J}\int_0^t R(t-s)f(s)ds. \tag{13}$$

We finish this section with a theorem from [6]. In this result, the constant D depends only on the fixed function $R(t)$ and the constant S. The theorem is still true when the functions are restricted to any interval $[0, K]$.

Theorem 2.1. *There is a constant $D > 0$ so that if $0 < t_2 < t_1 < \infty$, if $x \in BC$ with $|v(t, x(t))| \le S$, and if $0 < q < 1$ given in (1), then*

$$L := \left| \int_0^{t_1} R(t_1 - s)v(s, x(s))ds - \int_0^{t_2} R(t_2 - s)v(s, x(s))ds \right| \le D|t_1 - t_2|^q.$$

While it is easy to see that the function f is smoothed in passing to F, this is the first significant result indicating that the transformation from (5) to (12) has greatly smoothed the equation. We will show that by using a different norm the transformed function $v(t,x)$ will actually map bounded sets into compact sets. But if we are to use the new norm then we must show that our operations are still continuous in the new norm and that the contraction from $[(J-a(t))/J]x$ in the supremum norm will remain a contraction in the new norm. That is the task of the next section.

3. A weighted norm

We will define a certain Banach space $(\mathcal{B}, |\cdot|_g)$ and then from (12) defined a mapping by $\phi \in \mathcal{B}$ implies that

$$(P\phi)(t) = (A\phi)(t) + (B\phi)(t) \tag{14}$$

where

$$(A\phi)(t) = \frac{1}{J}\int_0^t R(t-s)v(s,\phi(s))ds \tag{15}$$

and

$$(B\phi)(t) = z(t) + F(t) + \int_0^t R(t-s)\left[\frac{(J-a(s))}{J}\phi(s)\right]ds. \tag{16}$$

As noted above, B is a contraction in the supremum norm in BC when F is bounded. We mention that if f is bounded, then F is bounded.

It turns out that continuity and compactness of the operator A on a bounded set M is greatly enhanced by working in the Banach space $(\mathcal{B}, |\cdot|_g)$ where $g : [0,\infty) \to [1,\infty)$ is continuous, $g(0) = 1$, $g \in\uparrow \infty$ as $t \uparrow \infty$, and $\phi \in \mathcal{B}$ if $\phi : [0,\infty) \to \Re$ is continuous and

$$|\phi|_g := \sup_{t\geq 0} \frac{|\phi(t)|}{g(t)} < \infty. \tag{17}$$

See [3, pp. 172-3] for properties of this space. We already know that if M is a set bounded in the supremum norm and if $A : M \to BC$ and is bounded in the supremum norm, then AM will be equicontinuous. If A is continuous on M in the supremum norm, then we change to the weighted norm. We now show that A is continuous on M in the weighted norm. Following that, we will refer to [3, pp. 172-3] to see that AM is contained in a compact set in that space.

Theorem 3.1. *Let $K > 0$ be given and let $M \subset \mathcal{B}$ be the closed set in $(\mathcal{B}, |\cdot|_g)$ of continuous functions $\phi : [0,\infty) \to \Re$ with $\|\phi\| \leq K$, $M = \{\phi \in \mathcal{B} | \|\phi\| \leq K\}$. If $A : M \to BC$ and is bounded in the supremum norm, say $\|AM\| \leq K^*$, then A is continuous on M in the norm $|\cdot|_g$.*

Proof. We actually prove uniform continuity. Thus, to say that A is continuous on M is to say that for each $\epsilon > 0$ there is a $\delta > 0$ such that $\phi, \eta \in M$ and $|\phi - \eta|_g < \delta$ implies that $|A\phi - A\eta|_g < \epsilon$. To show this continuity, let $\epsilon > 0$ be given.

There are two distinct parts. Notice that with $\phi, \eta \in M$ then $\|A\phi\|, \|A\eta\| \le K^*$ so for $T > 0$ with $g(T) > 2K^*/\epsilon$ we have

$$\sup_{t \ge T} \frac{|(A\phi)(t) - (A\eta)(t)|}{g(t)} \le \frac{2K^*}{g(T)} < \epsilon.$$

Thus, we need only work on $0 \le t \le T$.

Now $v(t, x)$ is continuous for $-K \le x \le K$ and $0 \le t \le T$ so it is uniformly continuous. Thus, for each $\epsilon > 0$ there is a $\delta g(T) > 0$ such that $|x|, |y| \le K$ and $|x - y| < \delta g(T)$ and $0 \le t \le T$ implies that $|v(t, x) - v(t, y)| < \epsilon$. Denote by $|\cdot|^{[0,T]}$ the supremum on $[0, T]$.

If $\phi, \eta \in M$ and $|\phi - \eta|_g < \delta$, then $|\phi(t) - \eta(t)|^{[0,T]} < \delta g(T)$ and

$$|v(t, \phi(t)) - v(t, \eta(t))|^{[0,T]} < \epsilon.$$

Hence,

$$\begin{aligned}
&\sup_{0 \le t \le T} \frac{|(A\phi)(t) - (A\eta)(t)|}{g(t)} \\
&= \sup_{0 \le t \le T} \frac{1}{g(t)} \int_0^t R(t-s)|v(s, \phi(s)) - v(s, \eta(s))|ds \\
&\le \sup_{0 \le t \le T} \frac{1}{g(t)} \int_0^t R(t-s)\epsilon ds \le \int_0^t R(t-s)\epsilon ds < \epsilon,
\end{aligned}$$

as required. □

If we change norm for the compact part of the mapping, then we must change it for the contraction part.

Theorem 3.2. *If $u(t, x) = [(J - a(t))/J]x$ satisfies a contraction condition for $\alpha < 1$ in the supremum norm (i.e., $|J - a(t)| \le \alpha J$), then B satisfies a contraction condition for the same constant α with respect to the g-norm.*

Proof. Let ϕ, $\psi \in \mathcal{B}$ and let $t > 0$ be arbitrary but fixed. Then

$$\begin{aligned}
&\left|\frac{(B\phi)(t) - (B\psi)(t)}{g(t)}\right| \\
&= \left|\int_0^t \frac{R(t-s)[u(s,\phi(s)) - u(s,\psi(s))]}{g(t)} ds\right| \\
&\le \int_0^t \frac{R(t-s)|u(s,\phi(s)) - u(s,\psi(s))|}{g(s)} ds \\
&\le \int_0^t R(t-s)ds \sup_{0\le s\le t} \frac{|u(s,\phi(s)) - u(s,\psi(s))|}{g(s)} \\
&\le \int_0^t R(t-s)ds \sup_{0\le s\le t} \frac{\alpha|\phi(s) - \psi(s)|}{g(s)} \\
&\le \int_0^t R(t-s)ds \sup_{t\ge 0} \frac{\alpha|\phi(t) - \psi(t)|}{g(t)} \\
&\le \alpha|\phi - \psi|_g.
\end{aligned}$$

This shows that we have a fixed upper bound so we can go back and take the supremum at each step. We have

$$\begin{aligned}
&\sup_{t\ge 0}\left|\frac{(B\phi)(t) - (B\psi)(t)}{g(t)}\right| \\
&= \sup_{t\ge 0}\left|\int_0^t \frac{R(t-s)[u(s,\phi(s)) - u(s,\psi(s))]}{g(t)} ds\right| \\
&\le \sup_{t\ge 0}\int_0^t \frac{R(t-s)|u(s,\phi(s)) - u(s,\psi(s))|}{g(s)} ds \\
&\le \sup_{t\ge 0} \frac{\alpha|\phi(t) - \psi(t)|}{g(t)} \\
&= \alpha|\phi - \psi|_g
\end{aligned}$$

as required. □

Our final theorem for this section can be found in [3, pp. 172-3], Example 3.1.6, but the proof is given here for convenience.

Theorem 3.3. *Let M be a closed, bounded, subset of BC and let AM be bounded in BC. The set AM is contained in a compact subset of $(\mathcal{B}, |\cdot|_g)$.*

Proof. We want to show that if $\{\phi_n\}$ is any sequence residing in AM then there is a subsequence converging to a continuous function ψ in the closure of AM in $(\mathcal{B}, |\cdot|_g)$. By repeated application of Ascoli's theorem there is a continuous function ψ contained in the closure of AM and there is a subsequence which converges uniformly to ψ on compact subsets of $[0,\infty)$. To see this, there is a subsequence denoted by $\{\phi_n^1\}$ converging uniformly to a continuous function ψ on $[0,1]$ since $\{\phi_n\}$

is uniformly bounded and equi-continuous on $[0,1]$ by Theorem 2.1. There is then a subsequence of this one, denoted by $\{\phi_n^2\}$, converging uniformly to a continuous function ψ on $[0,2]$ which coincides with the first named ψ on $[0,1]$. Continuing, we get a subsequence of each of the previous subsequences denoted by $\{\phi_k^k\}$ which converges uniformly on compact subsets to a continuous function which we call ψ on all of $[0,\infty)$.

Now, we want to show that $\{\phi_k^k\}$ converges to ψ in the g-norm. Let $\epsilon > 0$ be given and find N such that $2K^* < \epsilon g(N)$ where $\|AM\| < K^*$. Next, $\{\phi_k^k\}$ converges to ψ uniformly on $[0,N]$ since for $k > N$ this is a subsequence of $\{\phi_k^N\}$. Also $t \geq N$ implies that for any n we have

$$\frac{|\phi_n^n(t) - \psi(t)|}{g(t)} \leq \frac{2K^*}{g(N)} < \epsilon.$$

Thus we consider $\{\phi_n^n\}$ converging uniformly to ψ on $[0,N]$ and we take n large enough that

$$\frac{|\phi_n^n(t) - \psi(t)|}{g(t)} \leq |\phi_n^n(t) - \psi(t)| < \epsilon.$$

This completes the proof. □

4. An example

We first consider the sublinear case which can never be a contraction and, at the same time, illustrate the "built-in" perturbation. Notice that we ask $|x(0)| < 1 - (2/3\sqrt{3})$. In fact, it is understood that there is a perturbation, $D(t,x)$, so that the equation is actually ${}^cD^q x(t) = -x^{1/3}(t) + D(t,x)$. Once the set M is established for the unperturbed problem, then the same conclusion holds for the perturbed problem provided that for all $\phi \in M$ we have

$$|x(0)| + |D(t,\phi(t))| < 1 - \frac{2}{3\sqrt{3}}.$$

We are working with

$${}^cD^q x(t) = -x^{1/3}(t) + f(t), 0 < q < 1, \quad \|f\| \leq 1 - \frac{2}{3\sqrt{3}}, f \in C.$$

The system is in equilibrium at $t = 0$, but there is the persistent perturbation, $f(t)$, disturbing the equilibrium. Here are the details.

We will start out with $J = 1$ and then note that as $J \to 0$, we can pick up any $x(0)$ or bounded perturbation.

Example 4.1. If $|x(0)| < 1 - \frac{2}{3\sqrt{3}}$ then there is a solution of

$${}^cD^q x = -x^{1/3}(t), 0 < q < 1,$$

satisfying $|x(t)| < 1$ on $[0,\infty)$.

Proof. The equation corresponding to (12) is

$$x(t) = z(t) + \int_0^t R(t-s)[x(s) - x^{1/3}(s)]ds$$

and we need to find the maximum, say Q, of $|x - x^{1/3}|$ on some interval $[-L, L]$ with the property that $Q < L$. If we take

$$M = \{\phi : [0, \infty) \to \Re | \|\phi\| \le L\}$$

and take $|x(0)| \le L - Q$, it will then be true that $\phi \in M$ implies that $P\phi \in M$. If we take $L = 1$, then we find the maximum to be $Q = \frac{2}{3\sqrt{3}}$. And that solves the problem for $J = 1$. □

But the reader intuitively believes that we should obtain boundedness for any $x(0)$ and the intuition is correct. The solution resides in a choice of J and we proceed as follows. For an arbitrary $J > 0$ our equation (12) will have the integrand

$$x - (1/J)x^{1/3}$$

and that function vanishes at $x = 0$ and $x = J^{-3/2}$ with a maximum absolute value of

$$Q = \frac{2}{3\sqrt{3}J^{3/2}}.$$

on $[-L, L]$ with $L = J^{-3/2}$. We then need

$$|x(0)| \le J^{-3/2}\left(1 - \frac{2}{3\sqrt{3}}\right).$$

As $J \downarrow 0$, this value tends to ∞. In other words, any bounded perturbation will yield a bounded solution.

The interested reader may consult [6] for an example with $x^{1/3}$ replaced by x^3. That type of problem would not yield the kind of global results which we will seek in the coming sections.

5. An extension of Schaefer's theorem

The mappings we used in the foregoing examples are very effective for local results with highly nonlinear terms such as $v(t, x) = x^3$. But finding the mapping set M can be very tedious and restrictive; we could not, for example, allow F to be unbounded. If we seek global results with unbounded functions then Schaefer's fixed point theorem avoids all of that trouble, but adds something that is virtually never satisfied unless we trivialize the problem. If we work in $(\mathcal{B}, |\cdot|_g)$ and invoke Schaefer's theorem we are forced to ask that the mapping maps bounded sets into compact sets. But bounded sets in this space are enormous and so is their image under the mappings with which we are working.

Here is the idea we are developing. The setting is the Banach space $(\mathcal{B}, |\cdot|_g)$ where $|\cdot|_g$ is defined in (17) and $\phi \in \mathcal{B}$ if $\phi : [0, \infty) \to \Re$ is continuous and $|\phi|_g$ exists. A bounded set is contained in a ball of the form

$$\{\phi \in \mathcal{B}|\ |\phi|_g \leq D\}$$

for some $D > 0$. But if we restrict those functions to an arbitrary interval $[0, K]$ and map them according to Theorem 2.1 by a function

$$\int_0^t R(t-s)v(s, \phi(s))ds,$$

that truncated ball will be mapped into an equi-continuous set.

We are going to extend Schaefer's theorem to a general situation patterned after this mapping and we will use $|\cdot|_g$ to be definite, but many other norms would also work. Moreover, we will use contractions, but large contractions [2] can often be substituted so long as the nonlinearities are not too large.

Lemma 5.1. *Let $\eta : [0, \infty) \to [0, \infty)$ be continuous with $\eta(t) \to 0$ as $t \to \infty$. If $\{\phi_n\}$ is a sequence in $(\mathcal{B}, |\cdot|_g)$ with $|\phi_n(t)|/g(t) \leq \eta(t)$, and if $\{\phi_n\}$ is equi-continuous on any interval $[0, K]$, then there exists a subsequence $\{\phi_{n_k}\}$ of $\{\phi_n\}$ that converges to a function $\phi \in \mathcal{B}$, i.e., $|\phi_{n_k} - \phi|_g \to 0$ as $k \to \infty$, with $|\phi(t)|/g(t) \leq \eta(t)$ for all $t \geq 0$.*

Proof. We first observe that $|\phi_n(t)|/g(t) \leq \eta(t)$ for all $t \geq 0$ implies that $\{\phi_n\}$ is uniformly bounded on any compact subset of $[0, \infty)$. Since $\{\phi_n\}$ is equi-continuous on any interval $[0, K]$, we may repeatedly use Ascoli's theorem on intervals $[0, 1], [0, 2], ...$ to obtain a subsequence $\{\phi_{n_k}\}$ converging uniformly on any compact subsets of $[0, \infty)$ to a continuous function ϕ, i.e., $|\phi_{n_k}(s) - \phi(s)|^{[0,\ell]} \to 0$ as $k \to \infty$ for each fixed $\ell > 0$. For each $t \geq 0$ letting $k \to \infty$ in the following inequality

$$\begin{aligned} |\phi(t)|/g(t) &\leq |\phi(t) - \phi_{n_k}(t)|/g(t) + |\phi_{n_k}(t)|/g(t) \\ &\leq |\phi(t) - \phi_{n_k}(t)|/g(t) + \eta(t) \end{aligned}$$

we obtain $|\phi(t)|/g(t) \leq \eta(t)$ and so, $\phi \in \mathcal{B}$. Now let $\varepsilon > 0$ and choose $T > 0$ so that $\eta(t) < \varepsilon/2$ for all $t \geq T$. This yields

$$|\phi(t) - \phi_{n_k}(t)|/g(t) \leq 2\eta(t) < \varepsilon \ \text{ for } \ t \geq T.$$

Since $|\phi_{n_k}(s) - \phi(s)|^{[0,T]} \to 0$ as $k \to \infty$, there exists $N > 0$ such that $k \geq N$ implies that $|\phi(t) - \phi_{n_k}(t)| < \varepsilon$ for $0 \leq t \leq T$ and thus,

$$|\phi(t) - \phi_{n_k}(t)|/g(t) \leq |\phi(t) - \phi_{n_k}(t)| < \varepsilon$$

for all $0 \leq t \leq T$ and $k \geq N$. Combining the two inequalities above, we see that

$$|\phi_{n_k} - \phi|_g < \varepsilon \ \text{ for } \ k \geq N.$$

This shows $|\phi_{n_k} - \phi|_g \to 0$ as $k \to \infty$, and the proof is complete. □

In the proof of the next result we introduce sets denoted by G_n. Contrary to our geometrical intuition, it is true that they are closed, bounded, and convex which are necessary properties as we apply Schauder's theorem. The proof begins by following two lemmas of Rothe [17] and then follows the ideas of Schaefer [18] which are conveniently displayed in the book by Smart [19, pp. 26-32].

Theorem 5.2. *Let $T : \mathcal{B} \to \mathcal{B}$ be continuous and suppose that if $H \subset \mathcal{B}$ is bounded then TH is equi-continuous when the functions in H are restricted to any interval $[0, K]$. If, in addition, there exists a continuous function $\eta : [0, \infty) \to [0, \infty)$ with $\eta(t) \to 0$ as $t \to \infty$ such that $|(T\phi)(t)|/g(t) \leq \eta(t)$ for all $\phi \in H$, then either*

(i) *$x = \lambda Tx$ has a solution for $\lambda = 1$, or*
(ii) *the set of all such solutions, $0 < \lambda < 1$, is unbounded.*

Proof. For any $n > 0$ let

$$G_n = \{\phi \in \mathcal{B} |\ |\phi|_g \leq n\}.$$

First, there is a continuous map $r_n : \mathcal{B} \to \mathcal{B}$ with $r_n T\mathcal{B} \subset G_n$. Indeed, let $n > 0$ and $G = G_n$ be fixed. We denote by G^o the interior of G and by ∂G the boundary of G. Define the radial retraction $r : \mathcal{B} \to G$ by [19, pp. 26]

$$rx = \begin{cases} x & \text{if } x \in G \\ nx/|x|_g & \text{if } x \notin G \end{cases}$$

It is then true that
(i) r is a continuous retraction of $\mathcal{B}$ onto G,
(ii) if $rx \in G^o$ then $rx = x$,
(iii) if $x \notin G$ then $rx \in \partial G$.
From the definitions of rx and $|\cdot|_g$, we also see that $|(rx)(t)| \leq |x(t)|$ for all $x \in \mathcal{B}$ and $t \geq 0$.

With this in hand we go directly to the proof of Schaefer's theorem [19, pp. 29-30]. For any $n > 0$, if we choose r_n as the radial retraction of $\mathcal{B}$ onto G_n defined above, then the mapping $r_n T$ maps $\mathcal{B}$ into G_n. We drop the subscripts and call them r and G.

To see that rT is a compact map on G, let $\{\phi_m\}$ be any sequence in rTG, and let $\phi_m = rTx_m$ with $x_m \in G$. Now, $\{Tx_m\}$ is uniformly bounded and equi-continuous on every interval $[0, K]$. By the continuity of r, we see that $\{rTx_m\}$ is also equi-continuous on $[0, K]$ and

$$|\phi_m(t)|/g(t) = |rTx_m(t)|/g(t) \leq |Tx_m(t)|/g(t) \leq \eta(t)$$

for all $t \geq 0$. By Lemma 5.1, there exists a subsequence of $\{\phi_m\}$ that converges to a function $\phi \in \mathcal{B}$. Thus, the mapping rT is compact on G.

By Schauder's second theorem rT has a fixed point, x, in G. Either

(i) $|Tx|_g \le n$, in which case $Tx = rTx = x$, or
(ii) $|Tx|_g > n$, in which case $|x|_g = |rTx|_g = n$ so $x = rTx$ implies that $x = \frac{n}{|Tx|_g}Tx = \lambda Tx$ with $0 < \lambda < 1$.

Thus, for some (possibly large) n we get $Tx = x$ or the set of solutions $x = \lambda Tx$ is unbounded. □

The following example shows how the function η can be determined when T is in the form of mapping A in (15).

Example 5.3. Let $T : \mathcal{B} \to \mathcal{B}$ be defined by

$$(T\phi)(t) = \int_0^t R(t-s)\phi^{1/3}(s)ds.$$

For $\phi \in \mathcal{B}$ and $|\phi|_g \le L$, we have

$$\begin{aligned}|(T\phi)(t)|/g(t) &\le \int_0^t R(t-s)\left[|\phi(s)|/g(s)\right]^{1/3} ds/g^{2/3}(t)\\ &\le L^{1/3}/g^{2/3}(t) =: \eta(t).\end{aligned}$$

6. A Krasnoselskii-Schaefer fixed point theorem

Krasnoselskii studied a paper by Schauder on partial differential equations and formulated the following hypotheses for systems which are, in some sense, stable: The inversion of a perturbed differential operator yields the sum of a contraction and compact map. Accordingly he formulated a fixed point theorem taking both mappings into account. It was a combination of Banach's contraction mapping principle and Schauder's fixed point theorem. For Krasnoselskii's result, the aforementioned difficulty of setting up a self-mapping set, M, was even more pronounced than in the case of Schauder's theorem. There was certainly a need to avoid that difficulty and one way was given in [5] as follows.

Theorem 6.1. *Let $(\mathcal{B}, |\cdot|)$ be a Banach space, $A, B : \mathcal{B} \to \mathcal{B}$, B a contraction with constant $\alpha < 1$, and A continuous with A mapping bounded sets into compact sets. Either*
(i) *$x = \lambda B(x/\lambda) + \lambda Ax$ has a solution in $\mathcal{B}$ for $\lambda = 1$, or*
(ii) *the set of all such solutions $0 < \lambda < 1$ is unbounded.*

At this point it would be possible to relate the proof of that theorem and give a simple proof of an advance to the g-norm as a corollary to Theorem 5.2. However, it turns out that we can take ideas from [10] and remove the λ from the operator B. Here are the details.

Theorem 6.2. *Let $(\mathcal{B}, |\cdot|_g)$ be a Banach space, $A, B : \mathcal{B} \to \mathcal{B}$, B a contraction with contraction constant $\alpha < 1$, and A continuous. Assume* *that if $H \subset \mathcal{B}$ is bounded then AH is equi-continuous when the functions in H are restricted to any interval $[0, K]$. Finally, assume that there exists a continuous function $\eta : [0,\infty) \to [0,\infty)$ with $\eta(t) \to 0$ as $t \to \infty$ such that $|(A\phi)(t)|/g(t) \le \eta(t)$ for all $\phi \in H$. Either*

(i) *$x = Bx + \lambda Ax$ has a solution in $\mathcal{B}$ for $\lambda = 1$, or*
(ii) *the set of all such solutions, $0 < \lambda < 1$ is unbounded.*

Proof. For each positive integer n, let G_n and r_n be defined in the proof of Theorem 5.2. We choose n sufficiently large so that $B\emptyset \in G_n$ where $\emptyset$ is the zero element of $\mathcal{B}$. Now consider the mapping $(I-B)^{-1}r_nA$. For each $\phi \in \mathcal{B}$, we have $r_nA\phi \in G_n$. Moreover, if $y = (I-B)^{-1}r_nA\phi$, then $y = By + r_nA\phi$. We write $y = By - B\emptyset + B\emptyset + r_nA\phi$, and so

$$|y|_g \le |By - B\emptyset|_g + |B\emptyset|_g + |r_nA\phi|_g \le \alpha |y|_g + 2n.$$

Thus, $|y|_g \le 2n/(1-\alpha) =: n^*$. Define

$$G_{n*} = \{\phi \in \mathcal{B} |\; |\phi|_g \le n^*\}.$$

Since $(I-B)^{-1}$ is continuous, we see that $(I-B)^{-1}r_nA : G_{n*} \to G_{n*}$ is also continuous. Moreover, $(I-B)^{-1}r_nA$ is compact on G_{n*} by the proof of Theorem 5.2 and the continuity of $(I-B)^{-1}$. Thus, by Schauder's second theorem it has a fixed point, x, in G_{n^*}, i.e., $x = Bx + r_nAx$ (x depends on n). For some (possibly large) n, we have either

(i) $|Ax|_g \le n$, in which case $Ax = r_nAx$ and $x = Bx + Ax$, or
(ii) $|Ax|_g > n$ for all n, in which case $|r_nAx|_g = n$ and

$$r_nAx = \frac{n}{|Ax|_g}Ax =: \lambda Ax.$$

This yields $x = Bx + \lambda Ax$ and moreover

$$n = |r_nAx|_g = |x - Bx|_g \le (1+\alpha)|x|_g + |B\emptyset|_g.$$

Denote x by x_n. If (i) does not hold, then $|x_n|_g \to \infty$ as $n \to \infty$. This completes the proof. □

7. Examples

It may be disquieting to see the introduction of g in these problems without seeing it related to any of the terms in (2). We are now going to work through a series of three simple examples which are chosen to show easily and clearly what has been gained and the various roles that g can play in a problem and how the properties of g can be closely dictated by F and v. First, we will consider

$$x(t) = z(t) + \int_0^t R(t-s)\left[\frac{J - a(s))}{J}x(s) + \frac{v(s, x(s))}{J}\right]ds + F(t) \tag{12}$$

with $F(t)$ bounded. We will quickly show that there is a bounded solution and we will see that there is no need to specify $|\cdot|_g$ except to say g is continuous, $g(0) = 1$, and $g \in\uparrow +\infty$. When no g need be specified we get boundedness of solutions in the supremum norm. As we move to the next problems, g must be specified and it plays a central role.

Next, we will modify (12) by allowing $F(t)$ to become unbounded. This will force us to select a very definite $g(t)$ which will work into the equation in such a way that the boundedness work which we did in the first case will give us a solution which is bounded in the g-norm.

We start out with $v(t,x) = x^{1/3}$ in the first two problems, but for the third problem we take $v(t,x) = h(t)x^{1/3}$ where $h(t)$ is continuous and unbounded. Then a new g will be worked into the equation so that again we will use exactly the same argument to get a solution bounded in the new g-norm. We will also see that the modified $v(t,x)$ will map bounded sets into equi-continuous sets on truncated intervals $[0,K]$.

In these problems,

$$(Bx)(t) = z(t) + F(t) + \int_0^t R(t-s)\left[\frac{(J-a(s))}{J}\right]x(s)ds.$$

The remainder of the map defines the operator $(Ax)(t)$ with $v(t,x)$ being specified above. A few words should be said about the compactness of A. We discuss it for the third problem since it is most complicated. A bounded set $H \subset \mathcal{B}$ is typified by

$$\{\phi \in \mathcal{B} | |\phi(t)| \leq Lg(t), 0 \leq t < \infty\}.$$

The set AH is

$$AH = \{A\phi \in \mathcal{B} | (A\phi)(t) = \int_0^t R(t-s)h(s)\phi^{1/3}(s)ds,\ \phi \in H\}.$$

This is a set of functions bounded in the supremum norm when we truncate it by $0 \leq t \leq K$. Theorem 2.1 applies to it very well and it is equi-continuous when the functions in H are restricted to $[0,K]$.

By Theorem 6.2, to show that $B + A$ has fixed point in $\mathcal{B}$ it suffices to establish the existence of an *a priori* bound for all possible fixed points of $B + \lambda A$, $0 < \lambda < 1$. We illustrate this in three cases.

First case

First, if F is bounded, if $v(t,x)$ maps bounded sets in $(\mathcal{B}, |\cdot|_g)$ into bounded sets in this space, then the equi-continuity conditions of Theorem 6.1 are satisfied using Theorem 2.1. Specifically, suppose as we did earlier that $v(t,x) = x^{1/3}$ so that by the contraction condition in (4) and for arbitrary $x(0)$ we obtain an *a priori* bound on the solution of $x = Bx + \lambda Ax$ as follows. We have

$$|x(t)| \leq |z(t)| + \int_0^t R(t-s)[\alpha|x(s)|ds + \lambda|x^{1/3}(s)|]ds + |F(t)|$$

$$\text{(I)} \qquad \leq |x(0)| + \int_0^t R(t-s)[\alpha|x(s)| + |x^{1/3}(s)|]ds + |F(t)|$$

and both $|x(0)|$ and $|F|$ are fixed. Thus, we need only show that any solution of this inequality is bounded for a fixed bound which is independent of λ.

For the given $\alpha \in (0,1)$ there is a number $M > 0$ so that

$$|x^{1/3}| \leq M + \frac{(1-\alpha)}{2}|x|.$$

If we let $C_1 = |x(0)| + M + \|F\|$ then we have the fundamental relation which we will see three times

$$\text{(II)} \qquad |x(t)| \leq C_1 + \frac{1+\alpha}{2}\int_0^t R(t-s)|x(s)|ds.$$

It is then elementary to show that $x(t)$ is bounded and if X is either the maximum on an arbitrary interval $[0,U]$ or the supremum on $[0,\infty)$ of $|x|$ then

$$X \leq C_1 + \frac{1+\alpha}{2}X$$

or

$$\text{(III)} \qquad X \leq C_1\frac{2}{1-\alpha}.$$

This is the *a priori* bound and it is a bound in the supremum norm. That is the bound uniform in λ.

Now we consider $(A\phi)(t)$ where $\phi \in H$ and H is bounded. This means that there is a positive number L and if $\phi \in H$ then $|\phi(t)| \leq Lg(t)$. Thus,

$$\begin{aligned} |(A\phi)(t)|/g(t) &\leq \frac{\int_0^t R(t-s)|\phi(s)/g(s)|^{1/3}ds}{g^{2/3}(t)} \\ &\leq \frac{L^{1/3}}{g^{2/3}(t)} =: \eta(t). \end{aligned}$$

Thus, by Theorem 6.2, there is a solution of (12) on $[0,\infty)$ which is bounded as indicated with the bound depending only on $\alpha, |x(0)|, \|F\|$. Any g will suffice. It is only used in the compactness argument for Ax and that argument was described above concerning the third problem which is the most complicated.

Both of the other problems will be reduced to (I), (II), and (III) but x will be replaced by $x(t)/g(t)$ and the same bound will be in the g-norm.

Second case

We consider again the same equation with the only change being that $F(t)$ is unbounded. Clearly, no solution could be bounded in the supremum norm; if so, then the integral in (12) is bounded, leaving $F(t)$ free to be unbounded and taking $x(t)$ with it. But $g(t)$ will give us a growth condition on the solution. None of our work in Section 4 would handle such problems.

The weighting function g will be chosen as a continuous strictly increasing function $g(t)$ with $g(0) = 1$ and $|F(t)/g(t)| \leq K$ for some positive number K. With $|x(0)|$ fixed and x being the solution of $x = Bx + \lambda Ax, 0 < \lambda < 1$, we form

$$\frac{|x(t)|}{g(t)} \leq |x(0)| + K + \int_0^t R(t-s)\left[\frac{\alpha|x(s)|}{g(s)} + \left|\frac{x(s)}{g(s)}\right|^{1/3}\right]ds$$

which is parallel to (I) and we obtain an exact parallel to both (II) and (III). Our conclusion is that

$$\frac{|x(t)|}{g(t)} \leq C_1\frac{2}{1-\alpha}$$

or

$$|x|_g \leq C_1\frac{2}{1-\alpha}.$$

The same η argument works here.

Remark Now we see that our choice for g was arbitrary as far as the mapping T is concerned, just as in the former case. However, it is the forcing function now which demands that g be chosen so that F/g is bounded and, as F is unbounded, that will automatically select a g taking care of the mapping T.

Third case

Continue with the second case, but give $x^{1/3}(s)$ a coefficient $h(t)$ which is unbounded. Choose $g(t)$ as in the **Second case** except that

$$|F(t)/g(t)| + |h(t)/g^{2/3}(t)| \leq \Gamma$$

for some positive number Γ. Let x be the solution of $x = Bx+\lambda Ax, 0 < \lambda < 1$, and form the inequality as before

$$\frac{|x(t)|}{g(t)} \leq |x(0)| + \Gamma + \int_0^t R(t-s)\left[\frac{\alpha|x(s)|}{g(s)} + \frac{|h(s)x^{1/3}(s)|}{g(s)}\right]ds.$$

Notice that

$$\frac{h(s)x^{1/3}(s)}{g(s)} = \frac{h(s)x^{1/3}(s)}{g^{2/3}(s)g^{1/3}(s)} \leq \Gamma\left|\frac{x(s)}{g(s)}\right|^{1/3}.$$

Thus, our inequality becomes

$$\frac{|x(t)|}{g(t)} \leq |x(0)| + \Gamma + \int_0^t R(t-s)\left[\frac{\alpha|x(s)|}{g(s)} + \Gamma\left|\frac{x(s)}{g(s)}\right|^{1/3}\right]ds$$

which is again a perfect copy of (I) so that (II) and (III) will follow with a slight change in constants.

For the η argument, now go back and make g just a bit larger. Let $H \subset \mathcal{B}$ be bounded with $\phi \in H$ implying that $|\phi|_g \leq L$. We then have

$$|\phi(s)|^{[0,K]} \leq g(K)L.$$

and

$$|h(s)\phi^{1/3}(s)|^{[0,K]} \leq |h(s)|^{[0,K]}[g(K)L]^{1/3} =: L^*.$$

By Theorem 2.1, AH is equi-continuous when the functions in H are restricted to $[0, K]$.

The conditions of Theorem 6.2 are satisfied and we have global existence of a solution with a growth condition g.

Remark This is the first time we have seen the need for g based on the growth of the operator T. Thus, this g is doing double duty. It is providing a definite rate of growth to yield compactness and it is providing a growth rate for the solution. Again, our work in Section 4 would never have handled this kind of problem. Our new fixed point theorems have added a definite dimension to the scope of problems which can be treated.

8. Literature

Both Schaefer's and Krasnoselskii's results have been widely used for many decades. Recently there have been a number of contribution which relate to our work here.

We would first mention Park [16] who gives a very good summary of work related to Krasnoselskii's theorem and concludes with a contribution of his own which seems to be very general.

Our work has centered on fractional differential equations and weakening the compactness requirement. The examples show that the power of the results stems from the particular properties of the resolvent, $R(t)$. We have focused on the fact that weakening compactness demands strengthening continuity which is achieved mainly through the nice properties of $R(t)$. There are two papers which deal in depth with very general continuity and compactness questions. First, we would point to Barroso [1] and, more recently, Garcia-Falset [11].

All three of the aforementioned papers list numerous other references to Krasnoselskii's theorem. We are unaware of work in this context which has so completely focused on a weighted norm to achieve the compactness.

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Northwest Research Institute, 732 Caroline St., Port Angeles, WA
E-mail address: taburton@olypen.com

Department of Mathematics and Computer Science, Fayetteville State University, Fayetteville, NC 28301
E-mail address: bzhang@uncfsu.edu