FRACTIONAL DIFFERENTIAL EQUATIONS AND LIAPUNOV FUNCTIONALS

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Abstract. We consider a scalar fractional differential equation, write it as an integral equation, and construct several Liapunov functionals yielding qualitative results about the solution. It turns out that the kernel is convex with a singularity and it is also completely monotone, as is the resolvent kernel. While the kernel is not integrable, the resolvent kernel is positive and integrable with an integral value of one. These kernels give rise to essentially different types of Liapunov functionals. It is to be stressed that the Liapunov functionals are explicitly given in terms of known functions and they are differentiated using Leibniz's rule. The results are readily accessible to anyone with a background of elementary calculus.

1. Introduction

We study a fractional differential equation of Caputo type

$$ {}^{c}D^{q}x = f(t, x(t)), \quad 0 < q < 1, \tag{1} $$

with $f : [0, \infty) \times \Re \to \Re$ being continuous. Because it is of Caputo type it is inverted with simple initial conditions, just as an ordinary differential equation [16, p. 12], and written as

$$ x(t) = x(0) + \frac{1}{\Gamma(q)} \int_0^t (t - s)^{q-1} f(s, x(s))ds \tag{2} $$

where Γ is the gamma function. We refer the reader to Lakshmikantham, Leela, and Devi [16, p. 54] or to Chapter 6 of Diethelm [12, pp. 78, 86, 103] for proofs of the inversion. It is to be emphasized that (1) and (2) are not equivalent for the Riemann-Liouville derivative denoted by D^q instead of ${}^cD^q$ which is used in some of the papers listed below. In fact, Caputo introduced his derivative to avoid the initial conditions imposed by the Riemann-Liouville derivative which were difficult to reconcile with many real-world problems. Having made the point that (1) is of Caputo type, we now point out that there are initial conditions in the Riemann-Liouville problem which can sometimes be

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incorporated under the integral in (2) and $x(0)$ can be deleted. That will not be discussed further here.

Before the reader's patience wanes, let us quickly sketch what we accomplish here. The equation on which we focus is

$$x(t) = x(0) + F(t) - \frac{k}{\Gamma(q)} \int_0^t (t-s)^{q-1} g(s, x(s)) ds \tag{3}$$

where $x(0)$ is sometimes 0 and where either

$$xg(t,x) \geq 0 \text{ or } g(t,x) = x + G(t,x) \text{ with } |G(t,x)| \leq \phi(t)|x| \tag{4}$$

and ϕ is small in several different ways.

1. We construct a number of Liapunov functionals, stated for kernels of nonconvolution type, which are then readily applied to more general problems without pondering how to translate them.

2. These Liapunov functionals show that, loosely, there is a constant M so that if $F \in L^2[0,\infty)$ then

$$\int_0^t [g(s,x(s)) - \frac{\Gamma(q)}{k} F(s)]^2 ds \leq M \int_0^t F^2(u) du. \tag{5}$$

(See the end of the proof of Theorem 5.2.)

3. Next, we study the resolvent for the linear part of g. This gives a number of boundedness results for (2). Under a set of conditions we find that $|x(t) - F(t)| \to 0$ as $t \to \infty$. (See Cor. 6.4.)

Thus, 2. and 3. above give very precise information about the limit set of the solution and a measure of the speed to which the solution converges to its limit set.

4. Using the resolvent, R, in a Liapunov functional, several L^p results are obtained concerning (2).

This concludes the sketch and we return to discussion of (2). The literature is replete with applications when $q = 1/2$ and that value is a focus of Chapter 5 of Oldham and Spanier [22] in their treatment of fractional equations. With such wide application, it is of real concern to know that the solution of (2) does not experience drastic change as q passes through the value $1/2$. The concern is intensified when we note that $q = 1/2$ is a transitional value in the following sense:

(i) If $1/2 < q < 1$, then $t^{q-1} \in L^2[0,1]$, but $t^{q-1} \notin L^2[1,\infty)$.
(ii) If $0 < q < 1/2$, then $t^{q-1} \notin L^2[0,1]$, but $t^{q-1} \in L^2[1,\infty)$.
(iii) If $q = 1/2$, then $t^{q-1} \notin L^2[0,1]$ and $t^{q-1} \notin L^2[1,\infty)$.

The local L^2 integrability is an important property reflected throughout the entire early work of Tricomi [25]. Having raised the concern, let us hasten to state that all of our analysis fails to indicate serious instability as q passes through $1/2$.

Recent results and a variety of methods

The referee supplied the following references and we have found it to be an excellent selection because it gives a wide variety of applications and, perhaps more importantly, a variety of methods.

Uncertainties in differential equations and control theory have been studied for many years. An impressive theory has emerged from decades of work by the Krasovskii school in Ekaterinburg, Russia, among many other places. Such efforts seem to have been absent for fractional differential equations until introduced in a recent paper by Agarwal, Lakshmikantham, and Nieto [1]. They focus on a fuzzy fractional differential equation. An explicit example is given and a solution using the classical Mittag-Leffler function is obtained.

It turns out that there is more than one way to convert (1) to an integral equation. For example, an alternate way places only part of the equation under that integral sign which can be a great advantage when that integral introduces unboundedness which was not present in (1). Along an entirely different line, Banas and Zajac [2] introduce a Volterra-Stieltjes integral equation with a different set of advantages. Again, there is an instructive example.

Darwish and Ntouyas [11] use fractional integral equations to study radiative transfer and the kinetic theory of gases; their work centers on fixed point theory. In recent years there has been an increasing use of fixed point theory to show various intricate types of qualitative and stability properties [6]. Once the integral equation is derived, it can provide a convenient mapping into a variety of spaces so that a fixed point will inherit the properties of the space.

Zhang and Jiang [28] study a time-fractional axisymmetric diffusion-wave equation with a source term in cylindrical coordinates. They obtain an analytic solution using integral transforms which they study using numerical methods.

The present paper is totally different in every way from any of these examples. The purpose here is to study (2) by means of Liapunov's direct method. Equation (2) is a prime vehicle for demonstrating Liapunov's direct method for integral equations with a simple singularity in the kernel. It turns out that the kernel is completely monotone and that allows two lines of investigation. First, there is the natural Liapunov functional which can be derived from (2) by the simple sequence of squaring the equation, integration by parts, and application of the Schwarz inequality. In a sense to be described, it is a perfect Liapunov functional. Next, the resolvent, R, for the linear form of (2) is also completely monotone and, because the integral of the kernel diverges, it is true that $\int_0^\infty R(s)ds = 1$. This enables us to construct again a natural Liapunov functional for a perturbed form of (2) which is of a totally different type. Those two types of Liapunov functionals for singular kernels provide a broad introduction to Liapunov theory. We view this as an introduction and we spend considerable time explaining the process.

To be sure, Liapunov theory has been used for fractional equations for quite some time. The difference here is that our introduction is concrete, explicit, and very elementary. While the computations require careful attention, the mathematics is not beyond the reach of undergraduate mathematics majors. The Liapunov functionals consist of given elementary functions. The differentiation is a simple application of Leibniz's rule.

All of this is in marked contrast to the very deep and abstract Liapunov theory for fractional equations being developed by the Lakshmikantham school. For samples of that we refer the reader to the two prominent works [15] and [16] of Lakshmikantham, Leela, Sambandham, and Vasundhara Devi. That work is characterized by the relation

$$D^q V(t,x) = \limsup_{h\to 0} \frac{1}{h^q}[V(t,x) - V(t-h, x - h^q f(t,x))]. \tag{6}$$

where $V(t,x)$ is the Liapunov function. It is through that relation that the Liapunov function is related to the fractional differential equation. Our relation is totally different and elementary.

2. Examples of $(t-s)^{-1/2}$.

In many classical and modern problems involving partial differential equations and their integral equation counterparts there appears within the kernel the term $1/\sqrt{t-s}$. Chapter five of the book by Oldham and Spanier [22] is much concerned with this term resulting from $q = 1/2$. We mentioned the transitional value at $q = 1/2$ and, in view of the fundamental nature of the general set of problems studied at that value, there is reason to investigate the stability in a general sense of the results when q is near $1/2$. To put the study in context we will briefly describe a set of widely differing problems using $1/\sqrt{t-s}$ in their kernels.

In 1940 Consiglio [9] studied turbulance using

$$y(t) = (1/K)\left(1 - \int_0^t (t-s)^{-1/2} y^2(s)\,ds\right).$$

In 1951 Mann and Wolf [18] and Roberts and Mann [24] studied the temperature $u(x,t)$, in a semi-infinite rod by means of the integral equation

$$u(0,t) = \int_0^t \frac{G(u(0,s))}{\pi^{1/2}(t-s)^{1/2}}\,ds$$

where $G(1) = 0$, G is continuous, and $G(u)$ is strictly increasing. By a translation we would have a forcing function and a new G with $xG(x) \geq 0$, a condition we will see later. The problem was generalized and studied by Padmavally [23] in 1958 using a variable heating source described by

$$y(t) = \int_0^t \frac{G(s, y(s))}{\pi(t-s)^{1/2}} ds$$

under considerably more complicated conditions on G.

Miller [19, pp. 68– and 208–] continues that study dealing with an equation

$$y(t) = -(\pi K)^{-1/2} \int_0^t (t-s)^{-1/2} g(s, y(s)) ds$$

and a more general equation with forcing function under the assumption that $yg(t, y) \geq 0$. An elementary derivation of such equations is given by Weinberger [26, p. 357].

Nicholson and Shain [21] (see [19, p. 72-3]) study stationary electrode polography and obtain

$$x(bt) = e^{bt-u(t)} \left(1 - \int_0^{bt} (bt-s)^{-1/2} x(s) ds\right)$$

where $u(t)$ is given and b is a positive constant.

More recently, Kirk and Olmstead [13] studied blow-up in a reactive diffusive medium with a moving heat source. Their equation is

$$F(t, x_0(t)) = h(t) + \int_0^t \frac{e^{-\frac{[x_0(t)-x_0(s)]^2}{4(t-s)}}}{2(\pi(t-s))^{1/2}} ds.$$

Their study continues in [14].

Last, but by no means least, we mention the Abel equation,

$$f(t) = \int_0^t \frac{\phi(s)}{\sqrt{t-s}} ds,$$

a Volterra equation of the first kind. Tricomi [25, p. 39] remarks that Abel formulated and solved the equation in 1825, but did not realize the importance of the work. A clever application of the equation is found in the whale hunting saga of Herman Melville, "Moby Dick", Chapter XCVI.

Miller's assumption that $xg(t, x) \geq 0$ and the sign condition places the problem as the natural generalization of the stable linear equation and that is the approach we will take. We will arrive at an integral equation

$$x(t) = x(0) + F(t) - \frac{k}{\Gamma(q)} \int_0^t (t-s)^{q-1} g(s, x(s)) ds.$$

3. The general equation

We begin with a uniformly asymptotically stable linear equation and perturb it in several ways. At first we ask $x(0) = 0$ but this will change. The equation is

$$^cD^q x = f(t) - kg(t, x(t)), \quad x(0) = 0, \quad 0 < q < 1, \tag{7}$$

where k is a positive constant, $g : [0, \infty) \times \Re \to \Re$, $f : [0, \infty) \to \Re$ with both f and g continuous,

$$xg(t, x) \geq 0 \tag{8}$$

for $t \geq 0$ and $x \in \Re$. Define

$$F(t) := \frac{1}{\Gamma(q)} \int_0^t (t - s)^{q-1} f(s)ds. \tag{9}$$

Often we will ask that

$$\int_0^\infty F^2(t)dt < \infty, \tag{10}$$

but for resolvent theory we often ask F bounded, while in the final section we ask $F \in L^{2^p}$ for a positive integer p.

The Caputo equation (7) will be inverted as

$$x(t) = F(t) - \frac{k}{\Gamma(q)} \int_0^t (t - s)^{q-1} g(s, x(s))ds. \tag{11}$$

To relate this to the earlier discussion regarding physical problems, this is exactly the equation Miller [19, p. 209] considers and perturbs for his study in Theorem 6.1 on p. 210. Our methods and conclusions are far different than his.

When (10) is required, it is still true that F can be an arbitrarily large perturbation and it is persistent. We have $x(0) = 0$ and then we apply F. Condition (10) is easy to state and accept, but hard to understand. It forces conditions on q which we progressively improve as we work through the paper. At the end of the paper we give an appendix with some explanation. It is recommended that the reader review that information periodically.

4. Bounds on the singularity

In an earlier study [7] on singularity of convex kernels we noted that two properties are fundamental. When we construct a Liapunov functional for (11) it is critical that $F \in L^2[0, \infty)$. Moreover, when $g(t, x) = x$ then we study the resolvent equation by means of the Liapunov functional. To show the reader what must be done with a more general kernel, we will write $C(t, s)$ in place of $(t - s)^{q-1}$ and then restate the result in terms of this kernel.

The Liapunov theory which we developed in [7] requires positive constants α, β with $\alpha + \beta < 1$ and for some $\epsilon > 0$ then for $C(t,s) = (t-s)^{q-1}$ we must have

$$\int_s^{s+\epsilon} [\epsilon C_s(u, u-\epsilon) + C(u, u-\epsilon) + |C(u,s)|]du < \alpha \tag{12}$$

for $\epsilon \le s < \infty$ and

$$C(t, t-\epsilon)\epsilon + \int_{t-\epsilon}^{t} |C(t,s)|ds < \beta \tag{13}$$

for $\epsilon \le t < \infty$. The absolute value on C is needed since, in the general case, we have no sign specified at $t = s$. For the present problem that is not needed.

Much of the work in this section follows that in [7], as we said in the introduction, but here is a significant difference and a real simplification. We now show that these conditions automatically hold for our present problem.

Theorem 4.1. *If $C(t,s) = (t-s)^{q-1}$ where $0 < q < 1$ then for each fixed q there is an $\epsilon > 0$ for which (12) and (13) hold for $\alpha = \beta = 1/4$.*

Proof. Note that $C_s(u, u-\epsilon) = C_s(\epsilon) = (1-q)\epsilon^{q-2}$, $C(u, u-\epsilon) = \epsilon^{q-1}$, while $|C(u,s)| = (u-s)^{q-1}$. Thus, (12) is

$$\begin{aligned}
\int_s^{s+\epsilon} [(1-q)\epsilon^{q-1} + \epsilon^{q-1} + (u-s)^{q-1}]du &= (2-q)\epsilon^q + \frac{(u-s)^q}{q}\bigg|_s^{s+\epsilon} \\
&= (2-q)\epsilon^q + \frac{\epsilon^q}{q} \\
&= (2-q+\frac{1}{q})\epsilon^q < 1/4
\end{aligned}$$

if ϵ is sufficiently small and $q \in (0,1)$ is fixed.

Next, (13) is

$$\begin{aligned}
\epsilon^q + \int_{t-\epsilon}^{t} (t-s)^{q-1}ds &= \epsilon^q - \frac{(t-s)^q}{q}\bigg|_{t-\epsilon}^{t} \\
&= \epsilon^q + \frac{\epsilon^q}{q} \\
&= (1+\frac{1}{q})\epsilon^q < 1/4
\end{aligned}$$

for ϵ small and $q \in (0,1)$, a fixed number. □

5. Stability

Much of this section is a major application of [7] to the Volterra equation obtained from the fractional differential equation. It is an introduction to Liapunov theory for integral equations. Our first results rest on the Liapunov functional defined for some $\epsilon > 0$ and for $t \geq \epsilon$ by

$$V(t,\epsilon) = \int_0^{t-\epsilon} C_s(t,s)\bigg(\int_s^t H(u,x(u))du\bigg)^2 ds + C(t,0)\bigg(\int_0^t H(u,x(u))du\bigg)^2 \tag{14}$$

where

$$H(t,x(t)) = \frac{k}{\Gamma(q)} g(t,x(t)), \quad C(t,s) = (t-s)^{q-1}. \tag{15}$$

With C defined in this way and for $0 \leq s \leq t-\epsilon$, $t < \infty$, and $0 < q < 1$ we have

$$C(t,s) \geq 0, C_s(t,s) \geq 0, C_{st}(t,s) \leq 0, C_t(t,0) \leq 0. \tag{16}$$

As $s \leq t-\epsilon$ or $\epsilon \leq t-s$, both C_s and C_{st} are continuous.

Liapunov functionals often occur in a very natural way. This one is derived from (11) by squaring the equation, integrating by parts, and using the Schwarz inequality. Details are found in [5, p. 65] for a general convex kernel. (A version for integrodifferential equations was constructed by Levin [17] based on a suggestion of Volterra.) It is "perfect" in the sense that it will unite with the integral equation exactly with no inequalities required. The ϵ was introduced in [7] to avoid differentiating at the singularity, thereby introducing an error term which destroyed the perfection; however, Theorem 4.1 shows that the error tends to zero without changing the problem, restoring the perfection.

No chain rule

In the classical theory we have an ordinary differential equation, $x' = G(t,x)$, and we construct a Liapunov function, $V(t,x)$, which is totally unrelated to the differential equation. We may examine it and establish such properties as positive definite or radially unbounded without ever consulting $x' = G(t,x)$. The two are united as follows: If x is a soluton of $x' = g(t,x)$ then $W(t) := V(t,x(t))$ and

$$W'(t) = \frac{dW}{dt} = gradV \cdot G + \frac{\partial V}{\partial t}.$$

If V were so shrewdly constructed that $W'(t) \leq 0$ and if V is positive definite, then the zero solution is stable; if V is radially unbounded, then all solutions are bounded. Everything hinges on the chain rule and the independent properties of V. NONE OF THIS IS TRUE IN OUR WORK HERE! Our Liapunov functional is defined by (14). If $x : [0,\infty) \to \Re$ is ANY continuous function then we can differentiate $V(t,\epsilon)$ by Leibniz's rule. That is precisely what we will see through all the display sets in the proof of Theorem 5.1 until we come to the last display. That derivative of V will be correct for any continuous function, x. But, in particular, it is true when x solves (11); that equation is

$$\int_0^t C(t,s)H(s,x(s))ds = F(t) - x(t)$$

and the Liapunov functional was so shrewdly constucted that this integral occurs in the derivative of V and so we replace that integral by $F(t) - x(t)$. Now, the Liapunov functional is united to the integral equation and that derivative of the Liapunov functional is valid for the solution of the integral equation. No more can we say that the derivative is valid for any continuous function, x.

Those investigators familiar with the pioneering work done by Krasovskii [20] in the 1950s constructing Liapunov functionals for delay equations will recall that this was precisely the technique devised to convert the functional equation to a pointwise equation. That allowed Krasovskii to compare function to function, as opposed to the Razumikhin technique of comparing functions to functionals. Krasovskii's construction actually offered a way to extend Liapunov theory to integral equations as early as the mid 1950s, but neither he nor anyone else recognized it until 1992. It allows us to avoid the chain rule and the aforementioned supremum technique of the Lakshmikantham school.

The perfection of having exactly that last displayed integral occurring in the derivative of the Liapunov functional is very fruitful, but it is generally more than can be expected and much more than is needed. In this paper we will see a Liapunov functional united to an integral equation in three very different ways. A fourth way is seen in [7], proof of Theorem 5.1. A fifth way is seen in [5, pp. 62-63]. Historically, we have said that the aforementioned technique of Krasovskii is an art; precisely that same art is used here to replace the chain rule.

Theorem 5.1. *Let (8) and (16) hold. Let x be a continuous solution of (11) on $[0,\infty)$, let (14) be defined with this x, and let $\epsilon > 0$ be chosen so that (12) and (13) hold. Then*

$$\begin{aligned}\frac{dV(t,\epsilon)}{dt} &\leq 2H(t,x(t))\Big[F(t) - x(t) + C(t,t-\epsilon)\int_{t-\epsilon}^t H(u,x(u))du \\ &\quad - \int_{t-\epsilon}^t C(t,s)H(s,x(s))ds\Big] + C_s(t,t-\epsilon)\Big(\int_{t-\epsilon}^t H(u,x(u))du\Big)^2. \end{aligned} \tag{17A}$$

This is our general relation for integral equations.

Specializing to our problem for $C(t,s)$ and $H(t,x)$ yields

$$\begin{aligned}\frac{dV(t,\epsilon)}{dt} &\le (1-q)\epsilon^{q-2}\Big(\int_{t-\epsilon}^{t}\frac{k}{\Gamma(q)}g(u,x(u))du\Big)^2\\ &+2\frac{k}{\Gamma(q)}g(t,x(t))\Big[\epsilon^{q-1}\int_{t-\epsilon}^{t}\frac{k}{\Gamma(q)}g(u,x(u))du-\int_{t-\epsilon}^{t}(t-s)^{q-1}\frac{k}{\Gamma(q)}g(s,x(s))ds\Big]\\ &+2\frac{k}{\Gamma(q)}g(t,x(t))[F(t)-x(t)].\end{aligned} \tag{17B}$$

Factoring out $\frac{k^2}{\Gamma^2(q)}$ yields

$$\begin{aligned}\frac{dV(t,\epsilon)}{dt} &\le \frac{k^2}{\Gamma^2(q)}\Big[(1-q)\epsilon^{q-2}\Big(\int_{t-\epsilon}^{t}g(u,x(u))du\Big)^2\\ &+2g(t,x(t))\Big(\epsilon^{q-1}\int_{t-\epsilon}^{t}g(u,x(u))du-\int_{t-\epsilon}^{t}(t-s)^{q-1}g(s,x(s))ds\Big)\\ &+2g(t,x(t))[F(t)-x(t)]\frac{\Gamma(q)}{k}\Big].\end{aligned} \tag{17}$$

Proof. For $t \ge \epsilon$ we have $C_t(t,0) \le 0$ and $C_{st}(t,s) \le 0$ when $0 \le s \le t-\epsilon$ so by Leibniz's rule we have

$$\begin{aligned}V'(t,\epsilon) &\le C_s(t,t-\epsilon)\Big(\int_{t-\epsilon}^{t}H(u,x(u))du\Big)^2\\ &+2H(t,x(t))\int_0^{t-\epsilon}C_s(t,s)\int_s^t H(u,x(u))duds\\ &+2H(t,x(t))C(t,0)\int_0^t H(u,x(u))du.\end{aligned}$$

Integrating the next-to-last term by parts yields

$$\begin{aligned}&2H(t,x(t))\int_0^{t-\epsilon}C_s(t,s)\int_s^t H(u,x(u))duds\\ &=2H(t,x(t))\Big[C(t,s)\int_s^t H(u,x(u))du\Big|_0^{t-\epsilon}+\int_0^{t-\epsilon}C(t,s)H(s,x(s))ds\Big]\\ &=2H(t,x(t))\Big[C(t,t-\epsilon)\int_{t-\epsilon}^{t}H(u,x(u))du-C(t,0)\int_0^t H(u,x(u))du\\ &+\int_0^{t-\epsilon}C(t,s)H(s,x(s))ds\Big].\end{aligned}$$

Thus,

$$V'(t,\epsilon) \le C_s(t,t-\epsilon)\left(\int_{t-\epsilon}^{t} H(u,x(u))du\right)^2$$
$$+ 2H(t,x(t))\left[C(t,t-\epsilon)\int_{t-\epsilon}^{t} H(u,x(u))du + \int_0^{t-\epsilon} C(t,s)H(s,x(s))ds\right]$$
$$= C_s(t,t-\epsilon)\left(\int_{t-\epsilon}^{t} H(u,x(u))du\right)^2$$
$$+ 2H(t,x(t))\left[C(t,t-\epsilon)\int_{t-\epsilon}^{t} H(u,x(u))du + \int_0^{t} C(t,s)H(s,x(s))ds\right.$$
$$\left. - \int_{t-\epsilon}^{t} C(t,s)H(s,x(s))ds\right]$$

(Now, use (11) and trade $\int_0^t C(t,s)H(s,x(s))ds$ for $F(t)-x(t)$).

(Not only will it unite (11) and V, but will give us a negative definite term.)

$$= C_s(t,t-\epsilon)\left(\int_{t-\epsilon}^{t} H(u,x(u))du\right)^2$$
$$+ 2H(t,x(t))\left[C(t,t-\epsilon)\int_{t-\epsilon}^{t} H(u,x(u))du - \int_{t-\epsilon}^{t} C(t,s)H(s,x(s))ds\right]$$
$$+ 2H(t,x(t))[F(t)-x(t)]$$
$$= (1-q)\epsilon^{q-2}\left(\int_{t-\epsilon}^{t} \frac{k}{\Gamma(q)}g(u,x(u))du\right)^2$$
$$+ 2\frac{k}{\Gamma(q)}g(t,x(t))\left[\epsilon^{q-1}\int_{t-\epsilon}^{t} \frac{k}{\Gamma(q)}g(u,x(u))du - \int_{t-\epsilon}^{t}(t-s)^{q-1}\frac{k}{\Gamma(q)}g(s,x(s))ds\right]$$
$$+ 2\frac{k}{\Gamma(q)}g(t,x(t))[F(t)-x(t)]$$

as required. □

Three relations will be needed for us to parlay this Liapunov functional derivative into a qualitative result for a solution of (11). First, we must be able to estimate the relation between $xg(t,x)$ and $g^2(t,x)$. Our conditions (12) and (13) will allow such strongly singular kernels that it is not a great surprise to need $g(t,x)$ bounded by a linear function. We ask that

$$\frac{\Gamma(q)}{k}xg(t,x) \ge g^2(t,x). \tag{18}$$

Note that (18) defines sectors $\frac{\Gamma(q)}{k}|x| \ge |g(t,x)|$; as $q \downarrow 0$ the sector approximates the first and third quadrants. That property will vanish entirely when we study the resolvent and we do not have an interpretation of why that should occur.

Existence. Concerning existence theory, mild singularities such as $C(t,s) = (t-s)^{-p}$ for $0 < p < 1$ offer no existence problems at all when g is Lipschitz since in contraction mapping arguments $\int_0^t C(t,s)g(s,\phi(s))ds$ is continuous whenever ϕ is continuous. We use a weighted norm to get existence on an arbitrary interval $[0,T]$. See Becker [3] or Windsor [27] for simple and recent treatments of existence in the presence of such singularities. Miller [19] gives very general conditions for existence of solutions and there are other special existence results scattered throughout the literature. It would be a distraction and a limitation to repeat them here. To keep the focus on what is fundamentally new here, we simply work with problems in which a solution is known to exist.

Theorem 5.2. *Let x be a continuous solution of (11) on $[0,\infty)$ and let (16) and (18) hold. If, in addition, $F \in L^2[0,\infty)$ so are $g(t,x(t))$ and $g(t,x(t)) - \frac{\Gamma(q)}{k}F(t)$.*

Proof. Recall that we have shown that (12) and (13) hold. We begin by organizing the derivative of V which we computed in (17). First, by the Schwarz inequality we have

$$C_s(t,t-\epsilon)\left(\int_{t-\epsilon}^t H(u,x(u))du\right)^2 \le \epsilon C_s(t,t-\epsilon)\int_{t-\epsilon}^t H^2(u,x(u))du.$$

Next,

$$|2H(t,x(t))C(t,t-\epsilon)\int_{t-\epsilon}^t H(u,x(u))du| \le C(t,t-\epsilon)\int_{t-\epsilon}^t [H^2(t,x(t))+H^2(u,x(u))]du$$

and

$$|2H(t,x(t))\int_{t-\epsilon}^t C(t,s)H(s,x(s))ds| \le \int_{t-\epsilon}^t C(t,s)[H^2(t,x(t))+H^2(s,x(s))]ds.$$

Starting with (17A) we have

$$\begin{aligned}\frac{dV(t,\epsilon)}{dt} &\le 2H(t,x(t))[F(t)-x(t)] + C(t,t-\epsilon)\int_{t-\epsilon}^t (H^2(t,x(t)) + H^2(u,x(u)))du \\ &\quad + \int_{t-\epsilon}^t C(t,s)[H^2(t,x(t)) + H^2(s,x(s))]ds + \epsilon C_s(t,t-\epsilon)\int_{t-\epsilon}^t H^2(u,x(u))du\end{aligned}$$

or

$$\begin{aligned}\frac{dV(t,\epsilon)}{dt} &\le 2H(t,x(t))[F(t)-x(t)] + \left[\epsilon C(t,t-\epsilon) + \int_{t-\epsilon}^t C(t,s)ds\right]H^2(t,x(t)) \\ &\quad + \int_{t-\epsilon}^t [C(t,t-\epsilon) + C(t,s) + \epsilon C_s(t,t-\epsilon)]H^2(s,x(s))ds.\end{aligned} \tag{19A}$$

This is our general relation for integral equations. Specializing to our problem, we have from (15) that

$$\begin{aligned}\frac{dV(t,\epsilon)}{dt} &\leq \frac{2k}{\Gamma(q)}g(t,x(t))[F(t)-x(t)]\\&+\left[\epsilon\epsilon^{q-1}+\int_{t-\epsilon}^{t}(t-s)^{q-1}ds\right]\frac{k^2}{\Gamma^2(q)}g^2(t,x(t))\\&+\int_{t-\epsilon}^{t}[\epsilon^{q-1}+(t-s)^{q-1}+\epsilon(1-q)\epsilon^{q-2}]\frac{k^2}{\Gamma^2(q)}g^2(s,x(s))ds\end{aligned}$$

or

$$\begin{aligned}\frac{dV(t,\epsilon)}{dt} &\leq \frac{2k}{\Gamma(q)}g(t,x(t))[F(t)-x(t)]+[\epsilon^q+\frac{\epsilon^q}{q}]\frac{k^2}{\Gamma^2(q)}g^2(t,x(t))\\&+\int_{t-\epsilon}^{t}[\epsilon^{q-1}+(1-q)\epsilon^{q-1}+(t-s)^{q-1}]\frac{k^2}{\Gamma^2(q)}g^2(s,x(s))ds\end{aligned} \tag{19B}$$

This is our general expression for V' with our problem (11). There are many forms for the first term used here and we now make a special choice. From (18) we have

$$\begin{aligned}2g(t,x(t)[F(t)-x(t)]\frac{\Gamma(q)}{k} &\leq 2g(t,x(t))F(t)\frac{\Gamma(q)}{k}-2g^2(t,x(t))\\&=-g^2(t,x(t))-\left[g(t,x(t))-\frac{\Gamma(q)}{k}F(t)\right]^2+\frac{\Gamma^2(q)}{k^2}F^2(t).\end{aligned}$$

Using this in (19B) yields

$$\begin{aligned}\frac{dV(t,\epsilon)}{dt} &\leq \left[-g^2(t,x(t))-[g(t,x(t))-\frac{\Gamma(q)}{k}F(t)]^2\right]\frac{k^2}{\Gamma^2(q)}\\&+F^2(t)+\epsilon^q(1+\frac{1}{q})g^2(t,x(t))\frac{k^2}{\Gamma^2(q)}\\&+\int_{t-\epsilon}^{t}[\epsilon^{q-1}(2-q)+(t-s)^{q-1}]g^2(s,x(s))ds\frac{k^2}{\Gamma^2(q)}\end{aligned}$$

or

$$\begin{aligned}\frac{dV(t,\epsilon)}{dt} &\leq \frac{k^2}{\Gamma^2(q)}\left[-g^2(t,x(t))-[g(t,x(t))-\frac{\Gamma(q)}{k}F(t)]^2\right]+F^2(t)\\&+\left[\epsilon^q(1+\frac{1}{q})g^2(t,x(t))+\int_{t-\epsilon}^{t}[\epsilon^{q-1}(2-q)+(t-s)^{q-1}]g^2(s,x(s))ds\right]\frac{k^2}{\Gamma^2(q)}.\end{aligned}$$

We now innvoke (12) and (13) to find ϵ, α, and β and use Theorem 4.1 on the last line of that display. The first term is bounded by

$\beta g^2(t,x(t))$ so that we have

$$V'(t,\epsilon) \le \Big[\frac{\Gamma^2(q)}{k^2}F^2(t) - (g(t,x) - \frac{\Gamma(q)}{k}F(t))^2 - (1-\beta)g^2(t,x)$$

$$+\int_{t-\epsilon}^{t} [\epsilon^{q-1}(2-q) + (t-s)^{q-1}]g^2(s,x(s))ds\Big]\frac{k^2}{\Gamma^2(q)}. \tag{19}$$

In the last expression, integrate from ϵ to t, interchanging the order of integration by the Hobson-Tonelli test. We have

$$\begin{aligned}&\int_\epsilon^t \int_{u-\epsilon}^{u} [\epsilon^{q-1}(2-q) + (u-s)^{q-1}]g^2(s,x(s))dsdu\\ &\le \int_0^t \int_s^{s+\epsilon} [\epsilon^{q-1}(2-q) + (u-s)^{q-1}]du g^2(s,x(s))ds\\ &\le \alpha\int_0^t g^2(s,x(s))ds\end{aligned}$$

using (12) and Theorem 4.1. This now yields

$$\begin{aligned}V(t,\epsilon) \le V(\epsilon,\epsilon) + \Big[\int_\epsilon^t \frac{\Gamma^2(q)}{k^2}F^2(u)du - \int_\epsilon^t (g(s,x(s)) - \frac{\Gamma(q)}{k}F(s))^2 ds\\ -(1-\alpha-\beta)\int_\epsilon^t g^2(u,x(u))du + \alpha\int_0^\epsilon g^2(u,x(u))du\Big]\frac{k^2}{\Gamma^2(q)}.\end{aligned}$$

We assumed $x(t)$ exists so $V(\epsilon,\epsilon)$ is finite, while $V(t,\epsilon) \ge 0$. Put the negative terms on the left to finish the proof. □

Remark. In the last inequality we note that for M a positive constant we see an approximation to the relation

$$\int_\epsilon^t (g(s,x(s)) - \frac{\Gamma(q)}{k}F(s))^2 ds \le M\int_\epsilon^t \frac{\Gamma^2(q)}{k^2}F^2(u)du.$$

Thus, $g(t,x(t)) - \frac{\Gamma(q)}{k}F(t)$ gets small infinitely often. In Cor. 6.4 we see that $|x(t) - F(t)| \to 0$. We are acquiring very precise information about the solution of (11). In examining that relation, (9) shows that $\Gamma(q)$ is the denominator of F.

6. The resolvent and perturbations

Much of the last section was an application of the work in [7] for our fractional equation and a general introduction to Liapunov functionals for integral equations. But from here onward we leave that paper. Return to (7) and suppose that $x(0)$ is arbitrary, that $g(t,x) = x + G(t,x)$ where $G : [0,\infty) \times \Re \to \Re$ is continuous, and that there is a continuous function $\phi : [0,\infty) \to [0,\infty)$ with

$$|G(t,x)| \le \phi(t)|x|. \tag{20}$$

Our counterpart of (11) is

$$x(t) = x(0) + F(t) - \frac{k}{\Gamma(q)} \int_0^t (t-s)^{q-1}[x(s) + G(s, x(s))]ds. \tag{21}$$

Denote

$$C(t) = \frac{k}{\Gamma(q)} t^{q-1} \tag{22}$$

so that for any $T > 0$ we have the critical property that

$$\int_0^T |C(u)|du < \infty. \tag{23}$$

We now refer to a lengthy study detailed in Miller [19] extending from p. 193 to p. 222. For us it begins on p. 221 where it is noted that for $C(t)$ defined in (22) then

$$C(t) \text{ is completely monotone on } (0, \infty) \tag{24}$$

in the sense that $(-1)^k C^{(k)}(t) \geq 0$ for $k = 0, 1, 2, ...$ and $t \in (0, \infty)$. Moreover $C(t)$ satisfies the conditions of Miller's Theorem 6.2 on p. 212. That theorem states that if the resolvent equation for the kernel C is

$$R(t) = C(t) - \int_0^t C(t-s)R(s)ds \tag{25}$$

then that resolvent kernel, R, satisfies

$$0 \leq R(t) \leq C(t) \text{ for all } t > 0 \text{ so as } t \to \infty \text{ then } R(t) \to 0 \tag{26}$$

and that

$$C \notin L^1[0, \infty) \quad \Longrightarrow \int_0^\infty R(s)ds = 1. \tag{27}$$

Continuing on to pp. 221-224 (Theorem 7.2) we see that R is also completely monotone.

Next, under the conditions here, it is shown in Miller [19, pp. 191-207] that (21) can be decomposed into

$$y(t) = x(0) + F(t) - \int_0^t C(t-s)y(s)ds \tag{28}$$

and, having found $y(t)$, then the solution $x(t)$ of (21) solves

$$x(t) = y(t) - \int_0^t R(t-s)G(s, x(s))ds. \tag{29}$$

Something quite remarkable has happened. The kernel in (21) is not integrable on $[0, \infty)$, but in (29) it is replaced, not only by an integrable kernel, but the value of the integral is one and the new kernel is also completely monotone. Among the many useful properties that (29) has, we will see that the requirement (18) is not needed in (29). The term $\frac{k}{\Gamma(q)}$ has been absorbed and now is seen only in $F(t)$. A long line of results applies to (29), including the oldest and most elementary result in all of integral equation theory.

First, notice that the variation of parameters formula for (28) employs (25) and yields

$$y(t) = x(0) + F(t) - \int_0^t R(t-s)[x(0) + F(s)]ds. \tag{30}$$

From this we obtain several corollaries to (26) and (27).

Main note. In this section all of our conclusions, based on the use of the resolvent in the variation of parameters formula on the nonlinear equation, turn out to be of the L^∞ type. By contrast, in Section 8 all of our conclusions will be based on using R to form a Liapunov functional and those conclusions will be of the L^p type.

Corollary 6.1. *Let (26) and (27) hold. Then*

$$x(0) - \int_0^t R(t-s)x(0)ds \to 0, \tag{31}$$

$$F(t) \to 0 \text{ as } t \to \infty \text{ implies } \int_0^t R(t-s)F(s)ds \to 0, \tag{32}$$

$$\text{if } F \text{ is bounded so is } y, \tag{33}$$

and

$$\begin{aligned} &x(0) = 0 \quad \& F \in L^2[0,\infty) \implies \\ &\qquad y \in L^2[0,\infty) \text{ and } y(t) - F(t) \to 0 \text{ as } t \to \infty. \end{aligned} \tag{34}$$

Conclusions (31) and (33) are obvious, while (32) follows from the fact that the convolution of an L^1 function with a function tending to zero does, itself, tend to zero. To see (34), we have

$$\begin{aligned} (1/2)y^2(t) &\le F^2(t) + \left(\int_0^t R(t-s)F(s)ds\right)^2 \\ &\le F^2(t) + \int_0^t R(t-s)ds \int_0^t R(t-s)F^2(s)ds \end{aligned}$$

and that last integral is an L^1 function, being the convolution of two L^1 functions. (For reference in the appendix, if we square our inequality again we have y^4 and F^4.) In the same way, $(y(t) - F(t))^2 \le \int_0^t R(t-s)F^2(s)ds \to 0$ as $t \to \infty$

From (33) we obtain an elementary result which would not have been expected from examining (1) because $C \notin L^1[0,\infty)$.

Corollary 6.2. *If F is bounded and if there is an $\alpha < 1$ with*

$$\int_0^t R(t-s)\phi(s)ds \le \alpha$$

then $x(t)$ is bounded. This holds if $|\phi(t)| \le \alpha$.

Proof. We have $y(t)$ bounded by our remarks regarding (30). Thus, by way of contradiction, if x is not bounded then there is a sequence $\{t_n\} \uparrow \infty$ with $|x(t)| \le |x(t_n)|$ for $0 \le t \le t_n$ and $|x(t_n)| \uparrow \infty$. Then

$$|x(t_n)| \le \|y\| + \int_0^{t_n} R(t_n - s)\phi(s)|x(s)|ds \le \|y\| + \alpha|x(t_n)|,$$

a contradiction. □

If we had attempted this with (11), asking that $|g(t,x)| \le \alpha|x|$, we would have been stopped both by the magnitude of $k/\Gamma(q)$ and the divergence of $\int_0^t C(t-s)ds$. Both difficulties vanished because we transformed to (29).

Corollary 6.3. *Let $F(t)$ be bounded and suppose that*

$$\phi(t) = \phi_1(t) + \phi_2(t)$$

where $|\phi_1(t)| \le \alpha < 1$ and $\phi_2(t) \to 0$ as $t \to \infty$ or $\phi_2 \in L^1[0,\infty)$. Then $x(t)$ is also bounded.

Proof. By (33) $y(t)$ is bounded. Find $\{t_n\}$ as in the above proof and note that

$$\begin{aligned}|x(t_n)| &\le \|y\| + |x(t_n)| \int_0^{t_n} R(t_n - s)[|\phi_1(s)| + |\phi_2(s)|]ds \\ &\le \|y\| + |x(t_n)|[\alpha + \int_0^{t_n} R(t_n - s)|\phi_2(s)|ds].\end{aligned}$$

For either alternative, the integral tends to zero and so for large t_n that integral is smaller than $(1-\alpha)/2$. This is a contradiction. □

Corollary 6.4. *If y is bounded and if $\phi \in L^2[0,\infty)$, then $x(t) \to y(t)$ as $t \to \infty$. If, in addition, $F \in L^2[0,\infty)$ and $x(0) = 0$, then $x(t) \to y(t) \to F(t)$ as $t \to \infty$.*

Proof. We first show that x is bounded. By way of contradiction, if x is not bounded then find the sequence t_n as in the above proof so that

$$\begin{aligned}|x(t_n)| &\le \|y\| + |x(t_n)| \int_0^{t_n} R(t_n - s)\phi(s)ds \\ &\le \|y\| + |x(t_n)| \sqrt{\int_0^{t_n} R(t_n - s)ds \int_0^{t_n} R(t_n - s)\phi^2(s)ds}\end{aligned}$$

and the last term is the convolution of an L^1-function with a function tending to zero so it tends to zero, a contradiction to $|x(t_n)|$ being unbounded.

As $x(t)$ is bounded,

$$|x(t) - y(t)| \leq \|x\| \int_0^t R(t-s)\phi(s)ds$$
$$\leq \|x\| \sqrt{\int_0^t R(t-s)ds \int_0^t R(t-s)\phi^2(s)ds}$$

and this tends to zero, so $x(t) \to y(t)$ as $t \to \infty$. But by (34), $y(t) \to F(t)$. □

7. The Liapunov functional revisited

Consider now inequality (18) and notice that for fixed k, as $q \to 0$ then the condition on g relaxes, but as $k \to \infty$ then the condition becomes very severe. Note that in (28) Γ and k are in $F(t)$. As R is completely monotone, we could attempt to apply the Liapunov functional defined in (14) to (29), but there are two reasons why we would not want to do so in this particular problem. It could happen that this would be desired under other conditions so we note what must be done. First, we must ask that

$$xG(t,x) \geq G^2(t,x) \text{ and } y \in L^2[0,\infty)$$

and that is not desirable here because we take G to be a perturbation and we do not want to restrict it to $xG(t,x) \geq 0$. Next, the counterpart of Theorem 4.1 when C is replaced by R may be nontrivial. We leave it to the interested reader and move on to a new Liapunov functional which is consistent with the idea of G being more arbitrary.

8. Another Liapunov functional

The fact that $\int_0^\infty R(u)du = 1$ leads us to the possibility of new Liapunov functionals for (29) of the form

$$V(t,\epsilon) = \int_0^t \int_{t-s+\epsilon}^\infty R(u)du|G(s,x(s))|^p ds, \tag{35}$$

where p is some positive integer, in case

$$x(0) = 0 \text{ and } |G(t,x)| \leq \alpha|x|, \alpha < 1. \tag{36}$$

That Liapunov functional was introduced in [4] in a far more general form than is used here. It is an L^p counterpart of Corollary 6.2 which asks the unnatural property that F be bounded.

The following lemma is simple for this problem, but is a fundamental problem in the general case discussed in [4].

Lemma 8.1. *Let x solve (29) where R solves (25) so that (26) and (27) hold. Then for $\epsilon > 0$*

$$\sup_{0\leq s\leq t<\infty} \int_s^t |R(u+\epsilon-s) - R(u-s)|du \to 0$$

as $\epsilon \to 0$.

Proof. We can write the integral as

$$\int_0^{t-s} [R(v) - R(v+\epsilon)]dv \le \int_0^t [R(v) - R(v+\epsilon)]dv$$

by a change of variable and dropping the absolute values since R is positive and decreasing by the complete monotonicity. The supremum occurs as $t \to \infty$. For $t > 2\epsilon$ we have

$$\begin{aligned}
\int_0^t [R(v) - R(v+\epsilon)]dv &= \int_0^t R(v)dv - \int_0^t R(v+\epsilon)dv \\
&= \int_0^t R(v)dv - \int_\epsilon^{\epsilon+t} R(v)dv \\
&= \int_0^\epsilon R(v)dv + \int_\epsilon^t R(v)dv - \int_\epsilon^t R(v)dv - \int_t^{t+\epsilon} R(v)dv \\
&\le \frac{k}{\Gamma(q)} \int_0^\epsilon v^{q-1}dv - \int_t^{t+\epsilon} R(v)dv \\
&= \frac{k}{\Gamma(q)} \frac{v^q}{q}\Big|_0^\epsilon - \int_t^{t+\epsilon} R(v)dv \\
&= \frac{k}{\Gamma(q)} \frac{\epsilon^q}{q} - \int_t^{t+\epsilon} R(v)dv.
\end{aligned}$$

As $\epsilon \to 0$ the first term tends to zero. As the integral of R converges, the last term tends to zero as $t \to \infty$ for any fixed ϵ. □

The role of the coordinates t and s

We now offer a pair of results with an interesting connection to Corollary 6.2. Notice that we are integrating the kernel with respect to s, we are asking F bounded, and we are obtaining the solution, x, bounded. This is a form of the oldest result in integral equation theory. It is found, for example, in the classic book of Corduneanu [10, p. 127]. In [8] we argue that there are parallel theories of integral equations: integration of the s coordinate of the kernel is associated with a Razumikhin technique and yields solutions in L^∞, while integration of the t coordinate of the kernel is associated with a Liapunov technique and yields the solution in L^p. Thus, in the case of a convolution kernel, the integrations are equivalent and yield both the L^∞ result of Corollary 6.2 and the L^p results which we now display.

The first inequality in the proof of the next theorem can be iterated and the result is actually true for $y \in L^{2^p}$ and p is any positive integer. The details are discussed in the proof below and also in the appendix. The proof will show another way in which the Liapunov functional and the integral equation are united without using the chain rule. The appendix will show that in this theorem we need $0 < q < 1/2$, but as we let $p \to \infty$ we pick up the entire interval $0 < q < 1$.

Theorem 8.2. *Let x solve (29) so that (26) and (27) hold. Let (36) be satisfied. If $y \in L^2[0,\infty)$, so is $G(t,x(t))$.*

Proof. More work is required to unite the Liapunov functional with the integral equation. In (35) we now have $p = 2$ and we will contrive a derivative of V including the term $\int_0^t R(t-s)G^2(s,x(s))ds$. We must work with (29) to obtain that term. Here are the details. For any $\epsilon > 0$ we can find $M > 0$ so that

$$x^2(t) \le My^2(t) + (1+\epsilon)\Big(\int_0^t R(t-s)G(s,x(s))ds\Big)^2$$
$$\le My^2(t) + (1+\epsilon)\int_0^t R(t-s)ds\int_0^t R(t-s)G^2(s,x(s))ds$$

or

$$x^2(t) \le My^2(t) + (1+\epsilon)\int_0^t R(t-s)G^2(s,x(s))ds. \tag{37}$$

If we had $y \in L^4$ we would have taken $p = 4$ and we would have iterated the above process and obtained $\int_0^t R(t-s)G^4(s,x(s))ds$. (See the appendix. This will allow $0 < q < 3/4$.) For a general p, see [5, p. 62].

Let V be defined by (35) with $p = 2$ and obtain

$$V'(t,\epsilon) = \int_\epsilon^\infty R(u)duG^2(t,x(t)) - \int_0^t R(t+\epsilon-s)G^2(s,x(s))ds$$

(prepare to use (37))

$$\le G^2(t,x(t)) - \int_0^t R(t-s)G^2(s,x(s))ds$$
$$+ \int_0^t [R(t-s) - R(t+\epsilon-s)]G^2(s,x(s))ds$$

(use (37) and note that the sense of V' is unchanged)

$$\le G^2(t,x(t)) + \frac{1}{1+\epsilon}[My^2(t) - x^2(t)]$$
$$+ \int_0^t [R(t-s) - R(t+\epsilon-s)]G^2(s,x(s))ds,$$

as seen in the above display. But $x^2(t) \ge G^2(t,x(t))/\alpha^2$ where $\alpha < 1$, as seen in (36). This yields

$$V'(t,\epsilon) \le G^2(t,x(t)) + \frac{M}{1+\epsilon}y^2(t) - \frac{1}{\alpha^2(1+\epsilon)}G^2(t,x(t))$$
$$+ \int_0^t [R(t-s) - R(t+\epsilon-s)]G^2(s,x(s))ds.$$

But $\alpha < 1$ so we can take ϵ so small that $1 - \frac{1}{\alpha^2(1+\epsilon)} = -\mu$, for some $\mu > 0$.

If we integrate from 0 to t the last term is

$$\int_0^t \int_0^u [R(u-s) - R(u+\epsilon-s)]G^2(s,x(s))dsdu$$
$$= \int_0^t \int_s^t [R(u-s) - R(u+\epsilon-s)]duG^2(s,x(s))ds$$

by the Hobson-Tonelli test. For a sufficiently small ϵ by Lemma 8.1 we have $\int_s^t [R(u-s) - R(u+\epsilon-s)]du < \mu/2$. This yields

$$0 \le V(t,\epsilon) \le V(0,\epsilon) - \frac{\mu}{2}\int_0^t G^2(s,x(s))ds + \frac{M}{1+\epsilon}\int_0^t y^2(s)ds.$$

□

9. Another choice

Much of Sections 6 and 8 concerned the linear equation. The next section discusses the difficulties that arise with F in that treatment. We can avoid those by going directly to the nonlinear equation as follows; $y(t)$ will be zero. Changing notation, we consider

$$ {}^cD^q x = -kg(t,x(t)), \quad x(0) = 0, \quad 0 < q < 1, \tag{38}$$

with

$$g(t,x(t)) = x(t) + f(t) + G(t,x(t)) \tag{39}$$

so that

$$x(t) = -\frac{k}{\Gamma(q)}\int_0^t (t-s)^{q-1}[x(s) + f(s) + G(s,x(s))]ds. \tag{40}$$

Separate the equation as

$$y(t) = x(0) - \int_0^t \frac{k}{\Gamma(q)}(t-s)^{q-1}y(s)ds, \quad x(0) = 0, \tag{41}$$

and

$$x(t) = y(t) - \int_0^t R(t-s)[f(s) + G(s,x(s))]ds. \tag{42}$$

Notice that $y(t) = 0$ and define

$$L(t) = -\int_0^t R(t-s)f(s)ds. \tag{43}$$

If $f \in L^1$, so is L. If $f \in L^2$, so is L. We can go through a discussion parallel to the one given in Sections 6 and 8.

10. Appendix: $F, y, G \in L^{2p}$ and $0 < q < 1$

Several of our main results have asked for (10) to hold, while others ask for F to be bounded or tend to zero. We will discuss variations here closely related to statements in the proof of Theorem 8.2.

The function f in (7) is continuous and $0 < q < 1$ so for (9) the improper integral

$$\Gamma(q)F(t) = \int_0^t (t-s)^{q-1} f(s)ds$$

converges and so $F(t)$ is continuous. There is a progression of results. We will see that for (10) we need $0 < q < 1/2$. In the proof of Theorem 8.2 we discussed $y \in L^{2^p}$, requiring $F \in L^{2^p}$, and that will allow $0 < q < 1 - \frac{1}{2^p} \to 1$ as $p \to \infty$.

We start slowly. To see what is required for $F \in L^2[0, \infty)$ we study an example. Let T be an arbitrarily large positive number and let $f : [0, \infty) \to \Re$ be an arbitrary continuous function with the exception that $f(t) = 0$ on $[T-1, \infty)$.

For $t \geq T$ we have

$$\Gamma(q)F(t) = \int_0^{T-1} (t-s)^{q-1} f(s)ds = t^{q-1} \int_0^{T-1} (1 - \frac{s}{t})^{q-1} f(s)ds$$

and we see that on $[T, \infty)$ both $F(t)$ and the last integrand are continuous. Moreover, there is a positive number J with

$$\begin{aligned} \int_T^\infty (\Gamma(q)F(t))^2 dt &\leq J^2 \int_T^\infty t^{2(q-1)} dt \\ &= J^2 \frac{t^{2q-1}}{2q-1}\bigg|_T^\infty = \frac{J^2 T^{2q-1}}{1-2q} \leq \frac{J^2}{1-2q} \end{aligned}$$

since $0 < q < 1/2$.

We have learned that if we want (10) to hold for an arbitrary continuous $f \in L^p[0, \infty)$ for some $p \in [0, \infty]$, then a necessary condition is $0 < q < 1/2$. There is more to be learned. We noted after (9) that if $f \in L^1[0, \infty)$ then $F(t) \to 0$ as $t \to \infty$. Thus, we would expect that $F \in L^{2^p}[0, \infty)$ would relax $0 < q < 1/2$, and it does.

To see this, consider the remark after (37) in the proof of Theorem 8.2. If we need $G \in L^4$ then we must have $y \in L^4$ and we will obtain that by asking $F \in L^4$. (We repeat the process whereby we obtained (34).) A necessary condition for that conclusion is obtained from the calculation

$$\int_T^\infty t^{4(q-1)} dt < \infty$$

which holds if $0 < q < 3/4$.

Continuing as discussed in the proof of Theorem 8.2, for $G \in L^{2^p}$ we need $y \in L^{2^p}$ and so we need $F \in L^{2^p}$ which we can obtain if $0 < q < 1 - \frac{1}{2^p}$. That last term tends to one as p tends to infinity.

Thus, the methods here will finally offer necessary conditions to cover any q in the whole interval $0 < q < 1$.

Procedure for solving (7) with $x(0) = 0$, translated into (21). Fix $q \in (0, 1)$, let $f \in L^1[0, \infty)$, and find $F(t)$. For the given q, find 2^p so that $0 < q < 1 - \frac{1}{2^p}$. Verify that $F \in L^{2^p}$; our work yields a necessary condition, not a sufficient condition. Construct (35) with 2^p as the exponent of G and find $V'(t, \epsilon)$. Construct the counterpart of (37) with the square replaced by 2^p. Use it in the derivative of V to obtain the relations in the proof of Theorem 8.2. If $F \in L^{2^p}$, then we do get the same for y and G.

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References

[1] R. P. Agarwal, V. Lakshmikantham, J. J. Nieto, On the concept of solution for fractional differential equations with uncertainty, *Nonlinear Anal.:TMA*72(2010), 2859-2862.

[2] J. Banas and T. Zajac, A new approach to the theory of functional integral equations of fractional order, *J. Math. Anal. Appl.*375(2011), 375-387.

[3] L. C. Becker, Resolvents and solutions of weakly singular linear Volterra integral equations, *Nonlinear Anal.:TMA*74(2011), 1892-1912.

[4] L. C. Becker, T. A. Burton, and I. K. Purnaras, Singular integral equations, Liapunov functionals, and resolvents, submitted.

[5] T. A. Burton, *Liapunov Functionals for Integral Equations*, Trafford Publishing, Victoria, B. C., Canada, 2008 (www.trafford.com/08-1365).

[6] T. A. Burton, *Stability by Fixed Point Theory for Functional Differential Equations*, Dover, Mineola, New York, 2006.

[7] T. A. Burton, A Liapunov functional for a singular integral equation, *Nonlinear Anal.:TMA*, 73(2010), 3873-3882.

[8] T. A. Burton and J. R. Haddock, Parallel theories of integral equations, *Nonlinear Studies*, 17(2010), 177-197.

[9] A. Consiglio, Risoluzione di una equazione integrale non lineare presentatasi in un problema di turbalenza, *Academia Gioenia di Scienze Naturali in Cantania*, 4(1940), No. XX., 1-13.

[10] C. Corduneanu, *Differential and Integral Equations*, Chelsea, New York, 1977.

[11] M. A. Darwish and S. K. Ntouyas, On a quadratic fractional Hammerstein-Volterra integral equation with linear modification of the argument, *Nonlinear Analysis* 74(2011), 3510-3517.

[12] Kai Diethelm, *The Analysis of Fractional Differential Equations*, Springer, New York, 2004.

[13] C. M. Kirk and W. E. Olmstead, Blow-up in a reactive-diffusive medium with a moving heat source, *Zeitschrift ang. Mat. Physik ZAMP*53(2002)147-159.

[14] C. M. Kirk and W. E. Olmstead, Blow-up solutions of the two-dimensional heat equation due to a localized moving source, *Analysis and Appl.* 3(2005), 1-16.

[15] V. Lakshmikantham, S. Leela, and M. Sambandham, Lyapunov theory for fractional differential equations, *Communications in Applied Analysis* 12(2008), no. 4, 365-376.

[16] V. Lakshmikantham , S. Leela, J. Vasundhara Devi, *Theory of Fractional Dynamic Systems*, Cambridge Scientific Publishers, Cottenham, Cambridge, 2009.

[17] J. J. Levin, The asymptotic behavior of the solution of a Volterra equation, *Proc. Amer. Math. Soc.* 14(1963), 534-541.

[18] W. R. Mann and F. Wolf, Heat transfer between solids and gases under nonlinear boundary conditions, *Quat. Appl. Math.* 9(1951), 163-184.

[19] R. K. Miller, *Nonlinear Integral Equations*, Benjamin, Menlo Park, CA, 1971.

[20] N. N. Krasovskii, *Stability of Motion*, Stanford Univ. Press, 1963.

[21] R. S. Nicholson and I. Shain, Theory of stationary electrode polography, *Analytical Chemistry* 36(1964), 706-723.

[22] Keith B. Oldham and Jerome Spanier, *The Fractional Calculus*, Dover, New York, 2006.

[23] K. Padmavally, On a nonlinear integral equation, *J. Math. Mech.* 7(1958), 533-555.

[24] J. H. Roberts and W. R. Mann, On a certain nonlinear integral equation of the Volterra type, *Pacific J. Math.* 1(1951), 431-445.

[25] F. G. Tricomi, *Integral Equations*, Dover Publications, New York, 1985.

[26] H. F. Weinberger, *A First Course in Partial Differential Equations with Complex Variables and Transform Methods*, Blasidell, New York, 1965.

[27] A. Windsor, A contraction mapping proof of the smooth dependence on parameters of solutions to volterra integral equations, *Nonlinear Anal.:TMA* 72(2010), 3627-3634.

[28] F. Zhang and X. Jiang, Analytical solutions for a time-fractional axisymmetric diffusion wave equation with a source term, *Nonlinear Analysis:Real World Applications* 12(2011), 1841-1849.

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