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SIX INTEGRAL EQUATIONS AND A FLEXIBLE LIAPUNOV FUNCTIONAL

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Abstract

In this paper we study several integral equations including linear, nonlinear, and resolvent equations, by means of a flexible Liapunov functional. The goal is to obtain qualitative properties involving limit sets of solutions. The Liapunov functional is first applied directly to the integral equations without first differentiating the integral equation. In addition we develop a strategy for converting an integral equation into a strongly stable differential equation which maintains most of the properties of the kernel and then we apply that flexible Liapunov functional to it. None of this is applied to singular kernels, but work is in progress to apply the Liapunov functional to equations having singular kernels.

Key words: Liapunov functionals, Integral equations, Boundedness, Periodicity.

1. Introduction

We will be concerned with scalar integral equations of the forms

$$x(t) = a(t) - \int_{-\infty}^{t} C(t,s)g(s,x(s))ds, \; -\infty < t < \infty, \tag{1.1}$$

$$x(t) = a(t) - \int_{0}^{t} C(t,s)g(s,x(s))ds, \; 0 \le t < \infty, \tag{1.2}$$

$$R(t,s) = C(t,s) - \int_{s}^{t} C(t,u)R(u,s)du, \tag{1.3}$$

and

$$x(t) = a(t) - \int_{t-T}^{t} C(t,s)g(s,x(s))ds, \tag{1.4}$$

as well as two integrodifferential equations.

It is assumed that a, C, and g are continuous with

$$xg(t,x) \ge 0 \tag{1.5}$$

and

$$C \text{ is convex, as defined later.} \tag{1.6}$$

Much discussion of differential and integral equations with convex kernels can be found in [1–14, 16-17]. Many real-world problems are modeled with those kernels.

These problems have been studied since 1992 using Liapunov functionals, V, with $V'(t) \leq 2g(t,x)[a(t) - x(t)]$. The main difficulty is to write $V'(t) \leq -p(x(t)) + q(t)$ with p and q positive definite without severe conditions on g such as monotonicity. We introduce two techniques to counter those problems.

For our work here we will invoke existence theorems [5; pp. 166-180] showing that these equations have solutions which can be continued to $t = \infty$ if they remain bounded. We also refer to a theorem of Perron (see [1; p. 116]) which states that if $R(t,s)$ is continuous and if $\int_0^t R(t,s)a(s)ds$ is bounded for every bounded and continuous a, then $\sup_{0 \leq t < \infty} \int_0^t |R(t,s)|ds < \infty$.

Definition A function f is said to converge to a function g in $L^2[0,\infty)$ if $\int_0^\infty (f(t) - g(t))^2 dt < \infty$. A function f is said to converge to a function g pointwise if $|f(t) - g(t)| \to 0$ as $t \to \infty$.

2. A first Liapunov functional

We will treat (1.2) in some depth and then give brief treatments of the other equations. Write (1.2) as

$$x(t) = a(t) - \int_0^t C(t,s)g(s,x(s))ds \tag{2.1}$$

where

$$C(t,s) \geq 0, C_s(t,s) \geq 0, C_{st}(t,s) \leq 0, C_t(t,0) \leq 0 \tag{2.2}$$

and the subscript denotes the usual partial derivative. We introduce the theory by presenting one of the first Liapunov functional ever constructed for integral equations. Now, almost two decades later, it continues to be the source of a great many results.

Theorem 1. *Let (2.1) be a scalar equation and let (2.2) hold. Then along any solution of (2.1) the functional*

$$V(t) = \int_0^t C_s(t,s)\left(\int_s^t g(u,x(u))du\right)^2 ds + C(t,0)\left(\int_0^t g(s,x(s))ds\right)^2 \tag{2.3}$$

satisfies

$$V'(t) \leq 2g(t,x(t))[a(t) - x(t)]. \tag{2.4}$$

If there is a constant $B > 0$ with

$$C(t,t) \leq B \tag{2.5}$$

then along any solution of (2.1) we have

$$(a(t) - x(t))^2 \leq 2BV(t). \tag{2.6}$$

Proof.

Let us first see how we obtain the Liapunov functional, $V(t)$. We write down (2.1), square both sides, integrate by parts, separate the pieces in a standard way, then factor a term out of each piece. Here it is in symbols. We have

$$\begin{aligned}
(x(t)-a(t))^2 &= \Big(-\int_0^t C(t,s)g(s,x(s))ds\Big)^2\\
&= \Big(C(t,s)\int_s^t g(u,x(u))du\Big|_0^t - \int_0^t C_s(t,s)\int_s^t g(u,x(u))duds\Big)^2\\
&\le 2\Big[\Big(-C(t,0)\int_0^t g(u,x(u))du\Big)^2\\
&\quad+\Big(\int_0^t C_s(t,s)\int_s^t g(u,x(u))duds\Big)^2\Big]\\
&\le 2\Big[\Big(C(t,0)\int_0^t g(u,x(u))du\Big)^2\\
&\quad+\int_0^t C_s(t,s)ds\int_0^t C_s(t,s)\Big(\int_s^t g(u,x(u))du\Big)^2 ds\Big]\\
&\le 2\Big[C(t,0)+\int_0^t C_s(t,s)ds\Big]\Big[C(t,0)\Big(\int_0^t g(u,x(u))du\Big)^2\\
&\qquad+\int_0^t C_s(t,s)\Big(\int_s^t g(u,x(u))du\Big)^2 ds\Big]\\
&\le 2BV(t).
\end{aligned}$$

Remember, this is only true when x is a solution. Take $a(t)=0$ and notice that V is positive definite because of the solution x on the left. Also, if $a(t)$ is bounded, then $V\to\infty$ as $|x|\to\infty$ so V is radially unbounded, all because of the solution x. Unlike Liapunov functionals for differential equations, we can not determine that the Liapunov functional is radially unbounded without using the fact that x is a solution, as seen on the left-hand-side of this display.

Interestingly, while it is usually simple to differentiate a Liapunov function along the solution of a differential equation, it may be more difficult to differentiate the Liapunov functional along the solution of the integral equation than it is to construct the Liapunov functional. Note that if $x(t)$ is any continuous function then we can differentiate $V(t)$ by Leibnitz's rule and obtain

$$\begin{aligned}
V'(t) &= \int_0^t C_{st}(t,s)\Big(\int_s^t g(u,x(u))du\Big)^2 ds + 2g(t,x(t))\int_0^t C_s(t,s)\int_s^t g(u,x(u))duds\\
&\quad+C_t(t,0)\Big(\int_0^t g(s,x(s))ds\Big)^2 + 2g(t,x(t))C(t,0)\int_0^t g(s,x(s))ds.
\end{aligned}$$

We will see that one of these terms yields $\int_0^t C(t,s)g(s,x(s))ds$. It was constructed in that way, exactly as Krasovskii [11; p. 173-175] proceeded to construct Liapunov functionals in the 1950's. Thus, we will unite V and (2.1) by replacing that integral by its equivalent, $a(t)-x(t)$. To find that term, we integrate the third-to-last term by parts to obtain

$$\begin{aligned}
&2g(t,x(t))\Big[C(t,s)\int_s^t g(u,x(u))du\Big|_0^t + \int_0^t C(t,s)g(s,x(s))ds\Big]\\
&= 2g(t,x(t))\Big[-C(t,0)\int_0^t g(u,x(u))du + \int_0^t C(t,s)g(s,x(s))ds\Big]\\
&= 2g(t,x(t))[-C(t,0)\int_0^t g(u,x(u))du + a(t)-x(t)].
\end{aligned}$$

Collecting terms yields

$$V'(t) = \int_0^t C_{st}(t,s)\Big(\int_s^t g(u,x(u))du\Big)^2 ds + C_t(t,0)\Big(\int_0^t g(s,x(s))ds\Big)^2$$
$$+ 2g(t,x(t))[a(t)-x(t)] \le 2g(t,x(t))[a(t)-x(t)].$$

In (2.7) below it would gain us nothing to ask $|g(t,x)| \le K|x|$ instead of our $K=1$ for the K can be absorbed into $C(t,s)$ without changing the convexity. Here is our **first step**.

Theorem 2. *Let (1.5), (2.2), and (2.5) hold. Let*

$$|g(t,x)| \le |x| \tag{2.7}$$

so that

$$V'(t) \le -(a(t)-g(t,x))^2 - g^2(t,x) + a^2(t). \tag{2.8}$$

(i) If $a \in L^2[0,\infty)$ then $g(t,x(t)) \in L^2[0,\infty)$ and $g(t,x(t)) \to a(t)$ in $L^2[0,\infty)$.
(ii) If there is a $\Lambda > 0$ with

$$\int_0^t C_s(t,s)(t-s)\int_s^t a^2(u)du + C(t,0)t\int_0^t a^2(u)du \le \Lambda \tag{2.9}$$

then $(x(t)-a(t))^2$ is bounded. In particular, if

$$\int_0^t C_s(t,s)(t-s)^2 ds + C(t,0)t^2 \le \Lambda \tag{2.10}$$

then x is bounded for every bounded and continuous function a; if, in addition, $g(t,x)=x$ so that (2.1) is linear then by Perron's theorem $\sup_{t\ge 0}\int_0^t |R(t,s)|ds < \infty$ where R satisfies (1.3) and (3.3).

Proof. We have

$$V'(t) \le 2g(t,x)a(t) - 2g(t,x)x \le 2g(t,x)[a(t)-g(t,x)]$$
$$= -(a(t)-g(t,x(t))^2 - g^2(t,x(t)) + a^2(t)$$

so that an integration yields (i).

If $(x(t)-a(t))$ is unbounded then V is unbounded so there is a sequence $\{t_n\} \uparrow \infty$ with $V(t_n) \ge V(s)$ for $0 \le s \le t_n$ and $V(t_n) \uparrow \infty$. From this derivative relation we have for $0 \le s \le t_n$ that

$$0 \le V(t_n) - V(s) \le \int_s^{t_n} a^2(u)du - \int_s^{t_n} g^2(u,x(u))du.$$

If we take $t = t_n$ then

$$\int_s^t g^2(u,x(u))du \le \int_s^t a^2(u)du.$$

Thus for those t we have

$$\frac{1}{2B}(a(t)-x(t))^2 \le V(t)$$
$$= \int_0^t C_s(t,s)\Big(\int_s^t g(u,x(u))du\Big)^2 ds + C(t,0)\Big(\int_0^t g(u,x(u))du\Big)^2$$
$$\le \int_0^t C_s(t,s)(t-s)\int_s^t g^2(u,x(u))duds + C(t,0)t\int_0^t g^2(u,x(u))du$$
$$\le \int_0^t C_s(t,s)(t-s)\int_s^t a^2(u)duds + C(t,0)t\int_0^t a^2(u)du \le \Lambda$$

independent of n in t_n. This is a contradiction to V being unbounded. The rest is routine.

Now (2.4) is our fundamental relation and (2.7) is our **first step** in providing an effective separation of $x(t)$ and $a(t)$. The separation is perfect in that neither $g(t,x)$ nor $a(t)$ are modified in any way to establish the relation $\int_s^t g^2(s,x(s))ds \le \int_s^t a^2(s)ds$. Our **second step** will involve an integrodifferential equation and we must let $g(t,x) = g(x)$. **We will hereby see that Liapunov functionals for integral equations can be more general than Liapunov functionals for differential equations.**

3. Linear theory

We are going to show that for $g(t,x) = g(x)$ then equations

$$x(t) = a(t) - \int_0^t C(t,s)g(x(s))ds$$

and

$$y(t) = g^{-1}(a(t)) - \int_0^t C(t,s)y(s)ds$$

have solutions with similar limit sets. To put this in context we review some recent results.

When $g(t,x) = x$ then we are studying

$$x(t) = a(t) - \int_0^t C(t,s)x(s)ds \tag{3.1}$$

with resolvent equation

$$R(t,s) = C(t,s) - \int_s^t C(t,u)R(u,s)du \tag{3.2}$$

and variation of parameters formula

$$x(t) = a(t) - \int_0^t R(t,s)a(s)ds. \tag{3.3}$$

There are also perturbation formulae using the integral properties of R which we discuss extensively in [8]. In preparation for our work here, we review some properties found in [7]. First, for $V(t)$ defined by (2.3) we now have

$$V'(t) \le 2x(t)[a(t) - x(t)] = -(x(t) - a(t))^2 - x^2(t) + a^2(t) \tag{3.4}$$

so the separation is simple and very effective. If (2.5) holds then we have

$$\frac{1}{2B}(x(t) - a(t))^2 + \int_0^t (x(s) - a(s))^2 ds + \int_0^t x^2(s)ds \le \int_0^t a^2(s)ds. \tag{3.5}$$

If $a \in L^2[0,\infty)$ then $x \in L^2[0,\infty)$ and $x \to a$ in $L^2[0,\infty)$.

Theorem 3. *Let (2.2) and (2.5) hold so that (3.5) also holds. If $\int_0^t C_t^2(t,s)ds$ is bounded, if $a \in L^2[0,\infty)$, and if a is bounded, then x is bounded, $x \to a$ in $L^2[0,\infty)$ and pointwise.*

We have $\int_0^t (x(s) - a(s))^2 ds$ bounded and we can show that the derivative of the integrand is bounded by (3.5) and the Schwarz inequality used on $x'(t) - a'(t) = -C(t,t)x(t) - \int_0^t C_t(t,s)x(s)ds$. Thus, that integrand tends to zero.

Now, for the resolvent equation, $V(t)$ is readily changed to

$$W(t,s) := \int_s^t C_s(t,u)\left(\int_u^t R(v,s)dv\right)^2 du + C(t,s)\left(\int_s^t R(u,s)du\right)^2 \tag{3.6}$$

and we follow the computations for the proof of Theorem 1 to obtain the following result.

Theorem 4. *If $W(t,s)$ is defined in (3.6) then along a solution of (3.2) we have*

$$W_t(t,s) \le 2R(t,s)C(t,s) - 2R^2(t,s) = -(C(t,s)-R(t,s))^2 + C^2(t,s) - R^2(t,s), \quad 0 \le s \le t. \tag{3.7}$$

As $W(s)=0$ if (2.5) holds then we have

$$\frac{1}{2B}(R(t,s)-C(t,s))^2 \le W(t,s) \le -\int_s^t R^2(u,s)du + \int_s^t C^2(u,s)du - \int_s^t (C(u,s)-R(u,s))^2 du. \tag{3.8}$$

If, in addition, $\int_s^t [C^2(u,s) + C^2(t,u) + C_t^2(t,u)]du$ is bounded, then for fixed s $R(t,s) \to C(t,s)$ both pointwise and in $L^2[s,\infty)$.

It is a computation to establish that derivative of W and, hence, (3.8). The last stated boundedness conditions can be used with the resolvent equation and the Schwarz inequality to show that

$$C_t(t,s) - R_t(t,s) \text{ is bounded} \tag{3.9}$$

so convergence of $\int_s^t (C(u,s)-R(u,s))^2 du$ will imply that the integrand tends to zero.

With the strong convergence of $R(t,s)$ to $C(t,s)$ there arises the natural idea that perhaps we can replace $R(t,s)$ in the variation of parameters formula (3.3) with the given kernel $C(t,s)$. There is then the expectation that the error generated will be small.

Theorem 5. *If for fixed s we have $C(t,s) - R(t,s) \in L^2[s,\infty)$ and if $a \in L^1[0,\infty)$, then for*

$$y(t) = a(t) - \int_0^t R(t,s)a(s)ds$$

and

$$Y(t) = a(t) - \int_0^t C(t,s)a(s)ds$$

we have

$$y - Y \in L^2[0,\infty).$$

Moreover, if (3.9) holds then $y(t) \to Y(t)$ pointwise.

This concludes the review and we now return to Theorem 2 taking

$$g(t,x) = g(x),\ xg(x) \ge 0,\ |g(x)| \le |x|,\ \frac{dg(x)}{dx} > 0 \tag{3.10}$$

so that

$$V'(t) \le 2g(t,x)[a(t)-x(t)] \le 2g(x)[a(t)-g(x)] = -(a(t)-g(x))^2 - g^2(x) + a^2(t). \tag{3.11}$$

Modify (3.1) to

$$y(t) = g^{-1}(a(t)) - \int_0^t C(t,s)y(s)ds. \tag{3.12}$$

Theorem 6. *Let (2.2), (2.5), and (3.10) hold for (1.2). Suppose also that*

$$a \in L^2[0,\infty),\ g^{-1}(a(t)) \in L^2[0,\infty),\ \int_0^t C_t^2(t,s)ds \text{ is bounded.} \tag{3.13}$$

Then for the solution x of (1.2) we have

$$g(x(t)) \to a(t) \text{ in } L^2[0,\infty) \tag{3.14}$$

by Theorem 2, while the solution y of (3.12) satisfies

$$y(t) \to g^{-1}(a(t)) \text{ in } L^2[0,\infty) \text{ and pointwise} \tag{3.15}$$

if $a(t)$ is bounded by Theorem 3.

4. Strategy

Prior to 1992 the only way that Liapunov's direct method could be applied to integral equations was by first differentiating them. Miller [15] begins his Chapter 6 with this view and the work here begins at that point. Starting with $g(x)$ instead of $g(t,x)$ we have

$$x(t) = a(t) - \int_0^t C(t,s)g(x(s))ds. \tag{4.1}$$

If we assume that (2.2) holds and that a is differentiable then we can write

$$x'(t) = a'(t) - C(t,t)g(x(t)) - \int_0^t C_t(t,s)g(x(s))ds. \tag{4.2}$$

The problem is that for $C(t,s)$ convex, C_t is generally not convex. We are then faced with the necessity of working with a very small kernel, as in classical theory not involving Liapunov's direct method. There is a way out. Under some general conditions, if $C(t,s)$ is convex and if k is a sufficiently large constant, then

$$D(t,s) := C_t(t,s) + kC(t,s) \tag{4.3}$$

is convex; we have overwhelmed C_t. Thus, if we form

$$x'(t) + kx(t) = a'(t) + ka(t) - C(t,t)g(x(t)) - \int_0^t D(t,s)g(x(s))ds \tag{4.4}$$

then three things have happened. We have a differential equation together with a collection of Liapunov functions that go with it, when $xg(x) \geq 0$ and $C(t,t) \geq 0$ we have a perturbation of the uniformly asymptotically stable equation

$$x' + kx(t) + C(t,t)g(x(t)) = 0, \tag{4.5}$$

and we have a convex kernel.

The point of all this is that regardless of the value of k it is clear that a solution of (4.1) is also a solution of (4.4). Thus, if we can prove that all solutions of (4.4) enjoy a given qualitative property, then so does any solution of (4.1). The Liapunov functional used here is essentially that of Levin [12] and is discussed in Krasovskii [11; pp. 158-160].

In (i) below it would not be more general to ask $C(t,t) \geq \mu > 0$ instead of $C(t,t) \geq 1$ since μ could be absorbed into $g(x)$. We also ask that $a' + ka \in L^2[0,\infty)$, but we could avoid that and obtain boundedness following the proof of Theorem 2 with Λ.

Theorem 7. *If D is convex as in (2.2), if $xg(x) \geq 0$, and if*

$$L(t) = 2\int_0^x g(s)ds + \int_0^t D_s(t,s)\Big(\int_s^t g(x(u))du\Big)^2 ds + D(t,0)\Big(\int_0^t g(x(u))du\Big)^2 \tag{4.6}$$

then along any solution of (4.4) we have

$$L'(t) \leq 2g(x)[a'(t) + ka(t)] - 2g(x)[kx + C(t,t)g(x)]. \tag{4.7}$$

(i) Suppose that $C(t,t) \geq 1$. Then

$$L'(t) \leq -[a'(t) + ka(t) - g(x(t))]^2 - g^2(x(t)) - kxg(x) + [a'(t) + ka(t)]^2. \tag{4.8}$$

(ii) If, in addition, $a'(t)+ka(t) \in L^2[0,\infty)$, then $g(x(t)) \to a'(t)+ka(t)$ in $L^2[0,\infty)$, $\int_0^{x(t)} g(s)ds$ is bounded, $g(x(t)) \in L^2[0,\infty)$, and $xg(x) \in L^1[0,\infty)$.

Proof. We differentiate L by the usual chain rule along the solution of (4.4) and then integrate by parts as we did in the last part of the proof of Theorem 1 to obtain (4.7).

When (i) holds we have

$$L'(t) \leq 2g(x(t))[a'(t) + ka(t)] - 2g^2(x(t)) - 2kxg(x)$$

and (4.8) follows immediately. Next, integrate (4.8) and use $2\int_0^x g(s)ds \leq L(t)$ to get all of (ii).

Notice that it was a trade. Our separation in (4.8) requires nothing from g except sign, but we can not use (4.6) with $g(t,x)$.

5. The truncated equation

We consider the scalar equation

$$x(t) = a(t) - \int_{t-h}^t C(t,s)g(s,x(s))ds \tag{5.1}$$

where h is a positive constant,

$$|a(t)| < A \text{ for some } A > 0, \tag{5.2}$$

$$C(t,s) \geq 0, C_s(t,s) \geq 0, C_{st}(t,s) \leq 0, C(t,t-h) = 0, \tag{5.3}$$

and

$$xg(t,x) \geq 0. \tag{5.4}$$

Equation (5.1) can have three kinds of solutions. It might have a continuous solution on $(-\infty,\infty)$, possibly a periodic solution. It could have a continuous initial function $\phi : [t_0 - h, t_0] \to \Re$ and then have a solution $x(t)$ satisfying the equation for $t > t_0$ and $x(t) = \phi(t)$ on $[t_0 - h, t_0]$; if $\phi(t_0) = a(t_0) - \int_{t_0-h}^{t_0} C(t_0,s)g(s,\phi(s))ds$, then the solution is continuous, otherwise it is discontinuous. This is analogous to a functional differential equation having a solution with a cusp at t_0. It can be shown that for a given ϕ we may choose a different initial function arbitrarily close to ϕ so that the solution is continuous.

Theorem 8. *Let (5.3) and (5.4) hold and let x be any continuous solution of (5.1). Then the functional*

$$H(t) = \frac{1}{2}\int_{t-h}^t C_s(t,s)\left(\int_s^t g(v,x(v))dv\right)^2 ds \tag{5.5}$$

satisfies

$$(a(t) - x(t))^2 \leq 2H(t)C(t,t) \tag{5.6}$$

and

$$H'(t) \leq g(t,x(t))[a(t) - x(t)]. \tag{5.7}$$

Proof. We have

$$\begin{aligned} H'(t) = \quad & \frac{1}{2}\int_{t-h}^{t} C_{st}(t,s)\left(\int_{s}^{t} g(v,x(v))dv\right)^2 ds \\ & -\frac{1}{2}C_s(t,t-h)\left(\int_{t-h}^{t} g(v,x(v))dv\right)^2 \\ & + g(t,x(t))\int_{t-h}^{t} C_s(t,s)\int_{s}^{t} g(v,x(v))dv\,ds \\ & \le g(t,x(t))\int_{t-h}^{t} C_s(t,s)\int_{s}^{t} g(v,x(v))dvds. \end{aligned}$$

Integration of the last term by parts yields

$$\begin{aligned} & g(t,x(t))\left[C(t,s)\int_{s}^{t} g(v,x(v))dv\Big|_{s=t-h}^{s=t} + \int_{t-h}^{t} C(t,s)g(s,x(s))ds\right] \\ & \quad = g(t,x(t))\int_{t-h}^{t} C(t,s)g(s,x(s))ds. \end{aligned}$$

Then from (5.1) we have

$$\begin{aligned} H'(t) &\le g(t,x(t))\int_{t-h}^{t} C(t,s)g(s,x(s))ds \\ &= g(t,x(t))[a(t)-x(t)]. \end{aligned}$$

To obtain the lower bound on H we note that by the Schwarz inequality we have

$$\begin{aligned} \left(\int_{t-h}^{t} C_s(t,s)\int_{s}^{t} g(v,x(v))dv\,ds\right)^2 &\le \int_{t-h}^{t} C_s(t,s)ds\int_{t-h}^{t} C_s(t,s)\left(\int_{s}^{t} g(v,x(v))dv\right)^2 ds \\ &= 2H(t)C(t,t). \end{aligned}$$

Integration of the term on the left of this inequality by parts yields

$$\begin{aligned} & \left(C(t,s)\int_{s}^{t} g(v,x(v))dv\Big|_{s=t-h}^{s=t} + \int_{t-h}^{t} C(t,s)g(s,x(s))ds\right)^2 \\ & \quad = (a(t)-x(t))^2, \end{aligned}$$

as required.

Theorem 9. *Let (5.3) and (5.4) hold, let $|g(t,x)| \le |x|$, and let $C(t,t)$ be bounded. (i) If (5.2) holds then any continuous solution $x(t)$ of (5.1) is bounded and satisfies*

$$(x(t)-a(t))^2 \le h^2A^2C(t,t). \tag{5.8}$$

(ii) If $a \in L^2[0,\infty)$, then $g(t,x(t)) \to a(t)$ in $L^2[0,\infty)$.

Proof. From (5.7) we have

$$2H'(t) \le 2g(t,x(t))[a(t)-g(t,x(t))] = -(a(t)-g(t,x(t)))^2 - g^2(t,x(t)) + a^2(t).$$

Part (ii) follows from this.

If (5.2) holds and if $x(t)$ is not bounded then from (5.6) and $C(t,t)$ bounded we have $H(t)$ unbounded. Thus, there is a sequence $\{t_n\} \uparrow \infty$ with $H(t_n) \geq H(s)$ for $t_n - h \leq s \leq t_n$. For these s we have

$$0 \leq 2(H(t_n) - H(s)) \leq -\int_s^{t_n} g^2(u, x(u))du + \int_s^{t_n} a^2(u)du$$

or

$$\int_s^{t_n} g^2(u, x(u))du \leq hA^2.$$

By the Schwarz inequality

$$\begin{aligned} 2H(t_n) &\leq \int_{t-h}^t C_s(t,s)(t-s)\int_s^t g^2(v, x(v))dvds \\ &\leq h^2A^2 \int_{t-h}^t C_s(t,s)ds = h^2A^2C(t,s)\Big|_{t-h}^t \leq h^2A^2C(t,t) \end{aligned}$$

which is bounded, a contradiction. As $H(t)C(t,t)$ is bounded and a is bounded, from (5.6) we have x bounded so (i) is true.

Very often in problems in biology the functions are periodic because of life cycles or seasons changing. One of the most sought after properties is the existence of periodic solutions. It turns out that (5.8) almost gives us that free using Schaefer's fixed point theorem. It is discussed extensively in Chapter 5 of [5].

Theorem 10 Schaefer's Theorem. *Let $(X, \|\cdot\|)$ be a normed space, P a continuous mapping of X into X which is compact on each bounded subset of X. Then either*

(i) *the equation $x = \lambda Px$ has a solution in X for $\lambda = 1$, or*

(ii) *the set of all such solutions x, for $0 < \lambda < 1$, is unbounded.*

For (5.1) we take $(X, \|\cdot\|)$ to be the Banach space of continuous $T-$periodic functions with the supremum norm and we suppose that for this period $T > 0$ we have

$$C(t+T, s+T) = C(t,s),\ a(t+T) = a(t),\ g(t+T, x) = g(t,x). \tag{5.9}$$

Define P by $\phi \in X$ implies

$$(P\phi)(t) = a(t) - \int_{t-h}^t C(t,s)g(s, \phi(s))ds$$

and readily show that $(P\phi)(t+T) = (P\phi)(t)$ so that $P : X \to X$. It can be shown that P is continuous [5; p. 300].

Note that if Y is a fixed bounded subset of X and if ϕ is an arbitrary element of Y then

$$\frac{d}{dt}\int_{t-h}^t C(t,s)g(s,\phi(s))ds = C(t,t)g(t,\phi(t)) + \int_{t-h}^t C_t(t,s)g(s,\phi(s))ds$$

since $C(t, t-h) = 0$. This will be bounded over Y if

$$\int_{t-h}^t |C_t(t,s)|ds + C(t,t) \text{ is bounded.} \tag{5.10}$$

As a is uniformly continuous, Y is mapped into an equi-continuous set. This establishes the required compactness.

Theorem 11. *Let the conditions of Theorem 9 hold, as well as (5.9) and (5.10). Then (5.1) has a $T-$periodic solution.*

Proof. All of the conditions of Schaefer's theorem will be satisfied if we can show that there is a number M such that any solution of $x = \lambda Px$ for $0 < \lambda < 1$ satisfies $\|x\| \le M$. But all the work of Theorem 9 holds for the equation

$$x(t) = \lambda[a(t) - \int_{t-h}^{t} C(t,s)g(s,x(s))ds]$$

and by (5.8) we have the required bound.

6. Infinite delay

Consider the equation

$$x(t) = a(t) - \int_{-\infty}^{t} C(t,s)g(s,x(s))ds \tag{6.1}$$

with $g(t,x(t))$ bounded if $x(t)$ is bounded,

$$xg(t,x) \ge 0, \tag{6.2}$$

$$C(t,s) \ge 0, C_s(t,s) \ge 0, C_{st}(t,s) \le 0, C(t,t) \le B, \tag{6.3}$$

$$\lim_{s\to-\infty}(t-s)C(t,s) = 0 \text{ for each fixed } t, \tag{6.4}$$

and

$$\int_{-\infty}^{t} [C(t,s) + \big(C_s(t,s) - C_{st}(t,s)\big)(t-s)^2]ds \text{ is continuous .} \tag{6.5}$$

Equation (6.1) can have an initial function $\phi : (-\infty, t_0] \to R$; in that case, in order for the solution to be continuous at t_0 it is required that $\phi(t_0) = a(t_0) - \int_{-\infty}^{t_0} C(t_0,s)g(s,\phi(s))ds$. If ϕ is bounded and continuous, then (6.5) implies that this integral exists. Also, (6.1) may have a solution ψ on all of R; in that case, ψ is its own initial function on any interval $(-\infty, t_0]$.

Theorem 12. *Let (6.2) - (6.5) hold. If $x(t)$ is a continuous solution of (6.1) on an interval $[t_0,\infty)$ with bounded continuous initial function ϕ, then for*

$$H(t) = \int_{-\infty}^{t} C_s(t,s)\left(\int_{s}^{t} g(v,x(v))dv\right)^2 ds \tag{6.6}$$

we have

$$H'(t) \le 2g(t,x(t))[a(t) - x(t)]$$

and

$$(a(t) - x(t))^2 \le C(t,t)H(t). \tag{6.8}$$

Proof. A computation yields

$$\begin{aligned}
H'(t) =& \int_{-\infty}^{t} C_{st}(t,s)\left(\int_{s}^{t} g(v.x(v))dv\right)^2 ds + 2g(t,x(t)) \int_{-\infty}^{t} C_s(t,s) \int_{s}^{t} g(v,x(v))dvds \\
\le& 2g(t,x(t))\left[C(t,s)\int_{s}^{t} g(v,x(v))dv \Big|_{s=-\infty}^{s=t} + \int_{-\infty}^{t} C(t,s)g(s,x(s))ds\right] \\
=& 2g(t,x(t))[a(t) - x(t)]
\end{aligned}$$

where we have used the boundedness of the initial function (hence of $g(t,\phi(t))$) and (6.4) to conclude that the next to the last integral is zero. The existence of the first integrals shown in H' follows from (6.5).

For the lower bound on H we set

$$\begin{aligned}Q(t) :=&\Big(\int_{-\infty}^{t} C_s(t,s)\int_s^t g(v,x(v))dvds\Big)^2\\ \le&\Big(\int_{-\infty}^{t} C_s(t,s)ds\int_{-\infty}^{t} C_s(t,s)\Big(\int_s^t g(v,x(v))dv\Big)^2 ds\Big)\\ =&C(t,t)H(t).\end{aligned}$$

where we have taken $\lim_{s\to-\infty} C(t,s)=0$ using (6.4). On the other hand,

$$Q(t) =\Big(C(t,s)\int_s^t g(v,x(v))dv\,\Big|_{s=-\infty}^{s=t} + \int_{-\infty}^{t} C(t,s)g(s,x(s))ds\Big)^2 = (a(t)-x(t))^2.$$

Here is a very brief result which can be proved in a way parallel to Theorem 2.

Theorem 13. *Let the conditions of Theorem 12 hold, let $|g(t,x)| \le |x|$, let $a \in L^2[0,\infty)$, and let $x(t)$ be any continuous solution of (6.1) with bounded continuous initial function. Then $g(t,x(t)) \to a(t)$ in $L^2[0,\infty)$, $g(t,x(t)) \in L^2[0,\infty)$, while $a(t)$ and $C(t,t)$ bounded yields $x(t)$ bounded.*

For lack of space we do not include periodic theory. See [5; Chapters 5,6] and [15].

For the strategy, define D as in Section 4, let $g(t,x)=g(x)$, $C(t,t)\ge 1$, and form

$$x' = a'(t)+ka(t)-[kx(t)+C(t,t)g(x(t))]-\int_{-\infty}^{t} D(t,s)g(x(s))ds.$$

Then define

$$U(t)=2\int_0^x g(s)ds+\int_{-\infty}^{t} D_s(t,s)\Big(\int_s^t g(x(u))du\Big)^2 ds$$

and obtain a result parallel to that in Section 4.

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