PERIODICITY BY A PRIORI BOUNDS AND EXISTENCE

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Abstract. In this paper we define mappings for fixed point theory using the differential equation itself, rather than first changing to an integral equation. In the study of existence theory, the map defined here can smooth when the standard map does not; thus, we give the details of how to set up the mappings by proving two classical results. We then consider the equation $x' = p(t) - \int_{-\infty}^{t} C(t,s)g(x(s))ds$ and prove that there is a periodic solution using a fixed point theorem of Schaefer. Avoiding the integral equation can result in great simplification whenever g is monotone. We leave the reader with an interesting problem. In our proof we require that g be dominated by x^2. It would be so interesting to remove that condition as it seems to really have nothing to do with the basic problem; rather, it is used simply to make a proof work. Finally, we use the mappings and a fixed point theorem of Krasnoselskii to obtain a periodic solution of a neutral logistic equation $x'(t) = r(t, x'(t-h)) + g(x) + p(t)$. Here, r is not linear in x' and that makes the result of an essentially different type than those found in the literature.

1. Introduction and summary. In order to use fixed point theory to establish existence and qualitative properties of solutions of differential equations, it is customary to write the differential equation as an integral equation. The latter is then used to define a mapping with a fixed point which solves the problem.

a. Existence. A first example concerns the initial value problem

$$x' = f(t,x),\ x(t_0) = x_0,\ x \text{ and } f \in R^n. \tag{1.1}$$

We write

$$x(t) = x_0 + \int_{t_0}^{t} f(s,x(s))ds \tag{1.2}$$

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and define a mapping by

$$(P\phi)(t) = x_0 + \int_{t_0}^{t} f(s, \phi(s))ds. \tag{1.3}$$

If $P\phi = \phi$, then ϕ solves the problem. When f is continuous, then P is compact and Schauder's theorem yields a fixed point. If, in addition, f is Lipschitz in x, then P is a contraction with unique fixed point.

But when we come to neutral functional differential equations of the form

$$x'(t) = f(t, x(t), x'(t - h(t))) + g(t, x(t), x(t - h(t))) \tag{1.4}$$

where $0 \leq h(t) \leq h_0$ for some $h_0 > 0$, there are serious problems. If we ask that f and g be continuous, while f is Lipschitz in its second and third arguments, can we integrate (1.4) and say that

$$\int f(t, x(t), x'(t - h(t)))dt = F(t, x(t), x(t - h(t))) \tag{1.5}$$

where F is Lipschitz in its last two arguments? If we could say that, then an integration of (1.4) will yield a map

$$P\phi = B\phi + A\phi \tag{1.6}$$

where B is a contraction and A is compact; a fixed point theorem of Darbo or Krasnoselskii will yield a solution of (1.4). But investigators have been unable to verify (1.5) and, by technical necessity, have considered instead

$$\frac{d}{dt} D(t, x_t) = f(t, x_t) \tag{1.7}$$

where $x_t(s) = x(t + s)$, $s \leq 0$. An integration yields (1.6) and Darbo's theorem solves the problem (cf. Lakshmikantham et al [9] and Hale [7; p. 275]). The first idea of this paper concerns a different point of view. Instead of writing (1.1) as an integral equation, write a solution as an integral equation:

$$\Phi(t) = x_0 + \int_{t_0}^{t} \phi(s)ds. \tag{1.8}$$

Instead of using the mapping (1.3), let

$$(P\phi)(t) = f(t, \Phi(t)). \tag{1.9}$$

A fixed point ϕ yields Φ as a solution.

It turned out that mappings of the type in (1.9) can be compact when those of the type in (1.3) are not, as seen in [2]. Thus, the first part of the paper shows the details required in setting up a map of the (1.9) type by giving proofs of the standard existence theorem and uniqueness theorem for (1.1). In the last part of the paper we obtain a periodic solution of (1.4) using these mappings.

b. Periodicity. The search for periodic solutions of (1.1) adds two degrees of complication. First, we may believe that (1.1) has a periodic solution $x(t)$ when

$$f(t+T, x) = f(t, x), \tag{1.10}$$

but we have no way of knowing x_0; thus, we can not write (1.3). The second complication is that (1.3) must map some subset of $\mathcal{P}_T$, the set of continuous T-periodic functions, into itself. But unless $f(t, \phi(t))$ has mean value zero, then $P\phi$ in (1.3) will not be periodic. Investigators have addressed this problem in numerous ways and a selection of these may be seen in ([3], [4], [5], [10]). If, for example, (1.1) can be written as

$$x' = a(t)x + h(t, x) \tag{1.11}$$

where

$$a(t+T) = a(t), \;\; h(t+T, x) = h(t, x), \;\; \int_0^T a(t)dt < 0, \tag{1.12}$$

then we can use the variation of parameters formula to write (1.11) as

$$x(t) = \int_{-\infty}^{t} \left[\exp \int_s^t a(u)du\right] h(s, x(s))ds \tag{1.13}$$

and define

$$(P\phi)(t) = \int_{-\infty}^{t} \left[\exp \int_s^t a(u)du\right] h(s, \phi(s))ds. \tag{1.14}$$

Then $P : \mathcal{P}_T \to \mathcal{P}_T$ and both of the aforementioned complications are averted.

But (1.1) frequently does not fit in the neat form of (1.11). In the time-honored mathematical tradition of adding and subtracting the same thing, investigators have, on occasion (cf. [4]), written

$$x' = -x + x + f(t,x) \tag{1.15}$$

so that (1.13) becomes

$$x(t) = \int_{-\infty}^{t} e^{-(t-s)}[x(s) + f(s,x(s))]ds. \tag{1.16}$$

If f can dominate x in some way, then this may succeed. But in some of the most important problems f is bounded and (1.16) destroys the most useful properties of (1.1). Matters become much more difficult if we use degree-theoretic techniques. We describe these in the next few paragraphs.

Our first periodic result concerns the equation

$$x' = p(t) - \int_{-\infty}^{t} D(t,s)g(x(s))ds \tag{1.17}$$

with $g : R \to R$, g, p, and D continuous,

$$D(t,s) \geq 0, D_s(t,s) \geq 0, D_{st}(t,s) \leq 0, \tag{1.18}$$

and

$$p(t+T) = p(t), D(t+T, s+T) = D(t,s). \tag{1.19}$$

We studied this equation in [1] by means of a mapping like (1.9) when g is bounded and proved that there is a periodic solution without (1.18). Earlier [3], we studied the equation by means of a priori bounds when (1.18) holds by changing it first to an integral equation using essentially the idea of (1.15). Numerous difficulties were encountered and a very difficult boundedness argument was required. We show here that when g is increasing there is much simplification if a mapping like (1.9) is used. It is worth seeing some of the difficulties in the a priori bound arguments when we use the (1.15) technique.

If we use the method of a priori bounds and an integral equation for the mapping, then we need a homotopy equation for (1.17). Let $0 \le \lambda \le 1$ and write

$$x' + x = \lambda x - \lambda \int_{-\infty}^{t} D(t,s)g(x(s))ds + \lambda a(t). \tag{1.17$_\lambda$}$$

If x is a T-periodic solution of (1.17$_\lambda$), then multiply (17$_\lambda$) by e^t and integrate from $-\infty$ to t obtaining

$$x(t) = \lambda \int_{-\infty}^{t} e^{-(t-v)} \left[x(v) - \int_{-\infty}^{v} D(v,s)g(x(s))ds + p(v) \right] dv. \tag{1.20}$$

The mapping is then defined from (1.20) by

$$(H\varphi)(t) = \lambda \int_{-\infty}^{t} e^{-(t-v)} \left[\varphi(v) - \int_{-\infty}^{v} D(v,s)g(\varphi(s))ds + p(v) \right] dv. \tag{1.21}$$

If we were to use this method, then we would need to find an a priori bound on all fixed points of (1.21) for λ in $(0,1)$. As λ goes to 0, the added x in (1.17) completely changes the nature of the equation we started with. That will not happen when we use the method of (1.9).

We then use these mappings and a fixed point theorem of Krasnoselskii to obtain a periodic solution of a form of (1.4). Although much has been written about periodic solutions of (1.4), it is always assumed that f is linear in x'. Thus, our last result seems to be an essentially new type of contribution to the literature.

2. The basic mappings. We give two brief proofs which establish the technique needed for (1.9) in general existence theory. Classical treatment for Theorems 2.1 and 2.2 is found in Smart [13; pp. 4, 43, 44] and the interested reader can compare the difference

of the mapping sets for the two different methods. That difference is the main reason for giving these two proofs..

Let $x \in R^n$, $a > 0$, $b > 0$, and

$$\Omega = \{(t, x) \mid |t - t_0| \leq a, \ |x - x_0| \leq b\}. \tag{2.1}$$

Suppose that $f : \Omega \to R^n$ is continuous and consider the initial value problem

$$x' = f(t, x), \ x(t_0) = x_0. \tag{2.2}$$

THEOREM 2.1 (Cauchy-Picard). If f satisfies a Lipschitz condition on Ω with constant K and if $|f(t, x)| \leq M$ on Ω, then (2.2) has a unique solution for $|t - t_0| \leq \alpha$ where $0 < \alpha < \min[1/K, b/M, a]$. (By using a different metric than the standard one, $1/K$ can be deleted.)

PROOF. Let $(\mathcal{M}, \rho)$ be the complete metric space of continuous functions where

$$\begin{gathered} \mathcal{M} = \{\phi : [t_0 - \alpha, t_0 + \alpha] \to R^n \mid \phi(t_0) = f(t_0, x_0), \\ \|\phi\| \leq M, \ \phi \text{ continuous}\}, \\ \rho(\phi, \psi) = \|\phi - \psi\| = \sup_{|t - t_0| \leq \alpha} |\phi(t) - \psi(t)|, \end{gathered} \tag{2.3}$$

$|\cdot|$ is a norm on R^n and also denotes absolute value.

Here, our proof diverges from the classical one. For each $\phi \in \mathcal{M}$ define

$$\Phi(t) = x_0 + \int_{t_0}^{t} \phi(s) ds. \tag{2.4}$$

Thus, $\Phi(t_0) = x_0$, $|\Phi(t) - x_0| \leq M|t - t_0| \leq b$ and so $|f(t, \Phi(t))| \leq M$ since $(t, \Phi(t)) \in \Omega$. Then the mapping P defined on $\mathcal{M}$ by

$$(P\phi)(t) = f(t, \Phi(t)), \ |t - t_0| \leq \alpha, \tag{2.5}$$

is continuous, $(P\phi)(t_0) = f(t_0, x_0)$, and $|(P\phi)(t)| \leq |f(t, \Phi(t))| \leq M$. Hence, $P : \mathcal{M} \to \mathcal{M}$ and if $P\phi = \phi$, then Φ is a solution of (2).

Next, P is a contraction since

$$
\begin{aligned}
|(P\phi)(t) - (P\psi)(t)| &= |f(t,\Phi(t)) - f(t,\Psi(t))| \\
&\leq K|\Phi(t) - \Psi(t)| \\
&= \left| \int_{t_0}^{t} [\phi(s) - \psi(s)]ds \right| \\
&\leq K\alpha\|\phi - \psi\|
\end{aligned}
$$

and $K\alpha < 1$. Hence, there is a unique fixed point. Φ solves (2.2) and, clearly, is unique.

THEOREM 2.2 (Cauchy-Euler). Let Ω be defined in (2.1), $f : \Omega \to R^n$ be continuous, $|f(t,x)| \leq M$ on Ω, and $\alpha = \min[a, b/M]$. Then (2.2) has a solution for $|t - t_0| \leq \alpha$.

PROOF. Let $\mathcal{M}$, Φ, and P be defined in (2.3), (2.4), and (2.5). The set $\mathcal{M}$ is contained in a Banach space, $P : \mathcal{M} \to \mathcal{M}$, and P is continuous by the uniform continuity of f on Ω.

We now show that $P\mathcal{M}$ is equicontinuous so that Schauder's second theorem (cf. Smart [13; p. 25]) will yield a fixed point. Let $\epsilon > 0$ be given. We must find $\delta > 0$ so that $\phi \in \mathcal{M}$ and $|t_1 - t_2| < \delta$ imply that $|(P\phi)(t_1) - (P\phi)(t_2)| < \epsilon$. Now $|(P\phi)(t_1) - (P\phi)(t_2)| = |f(t_1,\Phi(t_1)) - f(t_2,\Phi(t_2))|$. Since f is uniformly continuous on Ω, for the given $\epsilon > 0$ there is a $\overline{\delta} > 0$ such that $|t_1 - t_2| < \overline{\delta}$ and $|\Phi(t_1) - \Phi(t_2)| < \overline{\delta}$ imply that $|f(t_1,\Phi(t_1)) - f(t_2,\Phi(t_2))| < \epsilon$. But $|\Phi(t_1) - \Phi(t_2)| = \left|\int_{t_1}^{t_2} \phi(s)ds\right| \leq M|t_2 - t_1| < \overline{\delta}$ provided that $|t_2 - t_1| < \overline{\delta}/(M+1) =: \delta$. This completes the proof.

3. Periodicity. This study began with

$$
x'(t) = -\int_{-\infty}^{t} C(t,s)g(x(s))ds + p(t) \tag{3.1}
$$

where C, g, and p are continuous and

$$
C(t+T, s+T) = C(t,s) \quad \text{and} \quad p(t+T) = p(t), \quad \text{some } T > 0, \tag{3.2}
$$

with a view to proving that there is a T-periodic solution. When C is of convolution type with compact support then (3.1) can take the form

$$
x' = -\int_{t-h}^{t} d(t-s)g(x(s))ds + p(t), \tag{3.3}
$$

about which much has been written. When $p(t) = 0$ it was used by Volterra [14] to model a population, while Levin and Nohel [11] use it to model a nuclear reactor, and Hale [7; pp. 120–3] points out that it can represent viscoelasticity. Again, when $p(t) = 0$ and when g satisfies some degenerate conditions, then both Levin and Nohel [11] and Hale [7; pp. 120–3] show that it can have periodic solutions.

For $p(t) = 0$, Levin introduced a Liapunov functional whose derivative was essentially $\int_{-\infty}^{t} C_{st}(t,s)(\int_s^t g(x(v))dv)^2 ds$. Our work here uses the idea from [3] that this integral can be used very effectively whenever

$$g(x)/x^2 \to 0 \text{ as } |x| \to \infty \tag{3.4}$$

and

$$-\int_{-\infty}^{t} \left[C_s^2(t,s)/C_{st}(t,s)\right] ds \le L \text{ for some } L > 0. \tag{3.5}$$

Now (3.5) is a mild condition as can be seen from $C(t,s) = (t-s+1)^{-3}$, in which case that integrand is $\frac{3}{4}(t-s+1)^{-3}$ and $L = \frac{3}{8}$. The Liapunov functional will require that

$$\int_{-\infty}^{t} \left[C(t,s) + C_s(t,s)(t-s)^2 + |C_{st}(t,s)|(t-s)^2\right] ds \tag{3.6}$$

be continuous and

$$\lim_{s \to -\infty} (t-s)C(t,s) = 0 \tag{3.7}$$

for fixed t. The mappings will need

$$\int_0^T p(s)ds = 0. \tag{3.8}$$

And from these conditions we can conclude that

$$\text{there is a } B > 0 \text{ with } \int_{-\infty}^{t} C_s(t,s)ds \le B. \tag{3.9}$$

Thus, we consider (3.1) with (3.2) and let

$$(\mathcal{P}_T^0, \|\cdot\|)$$

be the Banach space of continuous T-periodic functions with the supremum norm and having mean value zero. Note that if x is T-periodic and solves (3.1) then $x' \in P_T^0$. Thus, we define a map P on $\mathcal{P}_T^0$ by $\phi \in \mathcal{P}_T^0$ implies that

$$(P\phi_k)(t) = -\lambda \int_{-\infty}^{t} C(t,s) g\left(k + \int_0^s \phi(u)du\right) ds + p(t) \tag{3.10}$$

where k is a constant chosen so that $P\phi_k \in \mathcal{P}_T^0$. Conditions ensuring the existence of such a k are simple, natural, and consistent with traditional assumptions on (3.1).

PROPOSITION 3.1. Let (3.2) and (3.8) hold, $g* := dg/dx$ be continuous,

$$\int_{-\infty}^{t} |C(t,s)|ds \text{ be bounded,} \tag{3.11}$$

$$g(0) = 0, \quad g*(x) > 0, \tag{3.12}$$

and suppose that C is of one sign and not identically zero. Then there is a unique k so that $P\phi_k$ defined in (3.10) satisfies $P\phi_k \in \mathcal{P}_T^0$.

PROOF. As $g(x) > 0$ if $x > 0$, for any fixed $\phi \in \mathcal{P}_T^0$, there is a $\overline{k} > 0$ with $g\left(\overline{k} + \int_0^t \phi(s)ds\right) > 0$ and, in the same way, a $\overline{\overline{k}}$ with $g\left(-\overline{\overline{k}} + \int_0^t \phi(s)ds\right) < 0$ for all t. Now $\int_0^T \int_{-\infty}^t C(t,s) g\left(k + \int_0^s \phi(u)du\right) ds\, dt + \int_0^T p(s)ds$ is a continuous and monotone function of the constant k; moreover, it changes sign and so the required unique k is assured, yielding $P\phi_k \in \mathcal{P}_T^0$.

From here on, $P\phi_k = P\phi$.

PROPOSITION. 3.2. Let the conditions of Proposition 3.1 hold and for each $\phi \in \mathcal{P}_T^0$ pick that unique k and define P by (3.10). Then P is continuous.

PROOF. We will show that if $\phi \in \mathcal{P}_T^0$ is fixed and if $\phi_i \to \phi$, then $P\phi_i \to P\phi$. By way of contradiction, if $P\phi_i \not\to P\phi$, then there is a subsequence, say ϕ_i again, and $\delta > 0$ with $\|P\phi_i - P\phi\| \geq \delta$. As $\phi_i \to \phi$, it is clear that $\int_0^t \phi_i(s)ds \to \int_0^t \phi(s)ds$ so that if k and

k_i are the unique constants in the definitions of $P\phi$ and $P\phi_i$, then $k_i \not\to k$. In particular, there is a subsequence, say k_i again, and a $\mu > 0$ with $|k_i - k| \geq \mu$. Thus, for each $s \in [0,T]$ there is an $\eta(s)$ with

$$
\begin{aligned}
0 &= \int_0^T \int_{-\infty}^t C(t,s)\left[g\left(k + \int_0^s \phi(u)du\right) - g\left(k_i + \int_0^s \phi_i(u)du\right)\right] ds\, dt \\
&= \int_0^T \int_{-\infty}^t C(t,s)\left[g^*(\eta(s))\left(k - k_i + \int_0^s (\phi(u) - \phi_i(u))du\right)\right] ds\, dt
\end{aligned}
$$

and this is a contradiction since the right-hand-side is not zero when $\left|\int_0^t (\phi(u) - \phi_i(u))du\right| < \mu/2$. This completes the proof.

THEOREM 3.1. Let the conditions of Prop. 3.1 hold, let (3.2) and (3.4) - (3.9) hold, and let

$$
\int_{-\infty}^t |\partial C(t,s)\partial t| ds < \infty. \tag{3.13}
$$

Then (3.1) has a T-periodic solution.

PROOF. We first establish that there is a bound on all possible T-periodic solutions of

$$
x'(t) = \lambda[-\int_{-\infty}^t C(t,s)g(x(s))ds + p(t)]. \tag{3.1_λ}
$$

To that end, let x be such a solution and define a functional by

$$
V(t,x(\cdot)) = 2\int_0^x g(s)ds + \lambda \int_{-\infty}^t C_s(t,s)\left(\int_s^t g(x(v))dv\right)^2 ds \tag{3.14}
$$

so that along the T-periodic solution $x(t)$ of (3.1_λ) we have

$$
\begin{aligned}
V'(t,x(\cdot)) = \lambda &\int_{-\infty}^t C_{st}(t,s)\left(\int_s^t g(x(v))dv\right)^2 ds \\
&+ 2g(x)\left[-\lambda \int_{-\infty}^t C(t,s)g(x(s))ds + \lambda p(t)\right] \\
&+ 2\lambda g(x) \int_{-\infty}^t C_s(t,s) \int_s^t g(x(v))dv\, ds.
\end{aligned}
$$

If we integrate the last term by parts we get

$$2\lambda g(x)\int_{-\infty}^{t} C(t,s)g(x(s))ds$$

so that

$$V'(t,x(\cdot)) = \lambda\int_{-\infty}^{t} C_{st}(t,s)\Big(\int_{s}^{t} g(x(v))dv\Big)^2 ds + 2\lambda g(x)p(t). \tag{3.15}$$

Now for $\lambda > 0$ we have

$$\begin{aligned}
(\lambda p(t) - x'(t))^2 &= \Big(\lambda\int_{-\infty}^{t} C(t,s)g(x(s))ds\Big)^2\\
&= \Big(\lambda\int_{-\infty}^{t} C_s(t,s)\int_{s}^{t} g(x(v))dv\,ds\Big)^2\\
&= \Big[\lambda\int_{-\infty}^{t}\big[C_s(t,s)(-C_{st}(t,s))^{1/2}(-C_{st}(t,s))^{-1/2}\int_{s}^{t} g(x(v))dv\,ds\Big]^2\\
&\le -\lambda^2\int_{-\infty}^{t}\big[C_s^2(t,s)/C_{st}(t,s)\big]ds\int_{-\infty}^{t} -C_{st}(t,s)\Big(\int_{s}^{t} g(x(v))dv\Big)^2 ds\\
&\le -L\lambda^2\int_{-\infty}^{t} C_{st}(t,s)\Big(\int_{s}^{t} g(x(v))dv\Big)^2 ds.
\end{aligned}$$

Using this in (3.15) yields

$$V'(t,x(\cdot)) \le -\frac{1}{\lambda L}\Big(-x'(t)+\lambda p(t)\Big)^2 + 2\lambda p(t)g(x) \tag{3.16}$$

or

$$\begin{aligned}
\lambda L V'(t,x(\cdot)) &\le -(\lambda p(t) - x'(t))^2 + 2\lambda^2 L p(t)g(x(t))\\
&\le -(1/2)(x'(t))^2 + 5\lambda|p(t)|^2 + 2\lambda^2 L p(t)g(x(t)).
\end{aligned}$$

Since both x' and p are in $\mathcal{P}_T^0$ it follows that $\int_{-\infty}^{t} C(t,s)g(x(s))ds$ is also. This means that there is a t_1 in $[0,T]$ with $x(t_1)=0$. Also, there is a t_2 in $[0,T]$ with $\|x\| = |x(t_2)|$, say $0 \le t_1 < t_2 \le T$. Now

$$\Big|\int_{t_1}^{t_2} x'(s)ds\Big| = |x(t_2) - x(t_1)| = \|x\|$$

so

$$\|x\|^2 \le \left(\int_{t_1}^{t_2} |x'(s)|ds\right)^2 \le (t_2 - t_1)\int_{t_1}^{t_2} x'(s)^2 ds \le T\int_{t_1}^{t_2} x'(s)^2 ds.$$

Also, $V(T, x(\cdot)) = V(0, x(\cdot))$ and so an integration of V' yields

$$\begin{aligned} 0 &\le -(1/2)\int_0^T (x'(s))^2 ds + 2\lambda^2 L\int_0^T |p(t)||g(x(t))|dt \\ &\le -(1/2)\int_{t_1}^{t_2} (x'(s))^2 ds + 2\lambda^2 L\int_0^T |p(t)||g(x(t))|dt \\ &\le -(1/2T)\|x\|^2 + 2\lambda^2 L\int_0^T |p(t)||g(x(t))|dt \\ &\le -(1/2T)\|x\|^2 + 2L\|p\|\|g(x)\| \\ &\le 0. \end{aligned}$$

As g is increasing, $\|g(x)\| = |g(\pm\|x\|)|$. The last inequality and (3.4) yield an a priori bound on $\|x\|$. Schaefer's fixed point theorem (cf. Smart [12;p.29]) now shows that (3.10) has a fixed point ϕ in $\mathcal{P}_T^0$ for $\lambda = 1$ and the corresponding Φ is the T-periodic solution of (3.1). This completes the proof.

We now consider a neutral logistic equation

$$x'(t) = f(t, x'(t-h)) + ax(t) - bx^2(t) + p(t) \tag{3.17}$$

with $a > 0$, $b > 0$, $h > 0$,

$$|f(t,x) - f(t,y)| \le \alpha|x - y|, \quad \alpha < 1, \tag{3.18}$$

f and p continuous, and p satisfies (3.4). The object is to find conditions ensuring that (3.17) has a periodic solution. Extensive discussion is found in [6] and [8].

It is convenient to translate to a new origin by letting $y = a - bx$ so that $y' = -bx'$ and

$$y' = -bf\left(t, -\frac{1}{b}y'(t-h)\right) + y(y-a) - bp(t). \tag{3.19}$$

This equation has the general form of

$$y' = r(t, y'(t-h)) + g(y) + p(t) \tag{3.20}$$

where r is Lipschitz in y'. We need $r+g$ to map $\mathcal{P}_T^0$ into $\mathcal{P}_T^0$. Earlier [1], we had considered that equation under the assumption that it could be written as

$$y' = \alpha y'(t-h) + g(y) + p(t) \tag{3.21}$$

where g is continuous,

$$p \in \mathcal{P}_T^0, \quad T \leq 1, \tag{3.22}$$

and there are positive constants M_i with

$$g(0) = 0, \quad |g^*(y)| > 0 \text{ if } |y| \leq M_1, \tag{3.23}$$

$$|g(y)| \leq M_2 \text{ if } |y| \leq M_1. \tag{3.24}$$

Equation (3.21) is much easier to handle than (3.19) because we only need to deal with g since r will always have mean value 0. We proved the following result.

THEOREM. If (3.22)–(3.24) hold, and if $(|\alpha| M_1/2) + M_2 + \|p\| < M_1/2$, then (3.21) has a T-periodic solution.

Here, we will consider the full equation (3.19) and ask that r be continuous,

$$|r(t,x) - r(t,y)| \leq \alpha|x-y|, \; r(t,0) = 0, \tag{3.25}$$

$$p \in \mathcal{P}_T^0, \tag{3.26}$$

$$g(y) = y(y-a), a > 0, \tag{3.27}$$

and there is an $M > 0$ with

$$g(-MT + [a(1-\sqrt{2})/2]) + \|p\| \leq (1-3\alpha)M, M < a^2/2. \tag{3.28}$$

THEOREM 3.3. If (3.25) - (3.28) hold and if $a^2/4 > \alpha M + M^2T^2$, then ((3.20) has a T-periodic solution.

PROOF. The proof is based on the following fixed point theorem of Krasnoselskii which can be found in Smart [12;p. 31].

THEOREM (Krasnoselskii) Let $\mathcal{S}$ be a closed convex non-empty subset of a Banach space $(X, \|\cdot\|)$. Suppose that A and B map $\mathcal{S}$ into X such that

(i) $Ax + By \in \mathcal{S}(\forall x, y \in \mathcal{S})$,

(ii) A is continuous and $A\mathcal{S}$ is contained in a compact set,

(iii) B is a contraction with constant $\alpha < 1$.

Then there is a $y \in \mathcal{S}$ with $Ay + By = y$.

Thus, we let

$$\mathcal{S} = \{\phi \in \mathcal{P}_T^0 | \|\phi\| \leq M\}$$

and for $\phi \in \mathcal{S}$ define

$$(P_k\phi)(t) = r(t, \phi(t-h)) + g(k + \int_0^t \phi(s)ds) + p(t) \tag{3.29}$$

where k is a constant uniquely determined as follows.

First, for * denoting the derivative with respect to y, $g*(y) = 2y - a < 0$ if $y < a/2$. Also,

$$g(a/2) = -a^2/4 \text{ and } g(a(1-\sqrt{2})/2) = a^2/4. \tag{3.30}$$

Next, if $\phi \in \mathcal{S}$, by (3.21) we have

$$\max |r(t, \phi(t-h))| \leq \alpha M. \tag{3.31}$$

Clearly, $r(t, \phi(t-h)) \in \mathcal{P}_T$ and

$$|\int_0^T r(t, \phi(t-h))dt| \leq \alpha MT \text{ for } \phi \in \mathcal{S}. \tag{3.32}$$

Moreover, since $\int_0^T \phi(s)ds = 0$, it follows that

$$|\int_0^t \phi(s)ds| \leq MT/2.$$

Note that so long as $k + \int_0^t \phi(s)ds \leq a/2$ it follows that

$$G(k) := \int_0^T g(k + \int_0^t \phi(s)ds)dt$$

is monotone in k and continuous. We will find constants k_i so that the mean value of $P_{k_1}\phi$ is negative, while the mean value of $P_{k_2}\phi$ is positive. Thus the existence of the desired k will be assured. After establishing the existence of a unique k we will drop the subscript on P.

Let $\phi \in \mathcal{S}$ be fixed.

Case 1. $\int_0^T r(t, \phi(t-h))dt \leq 0$. Now, for

$$k = [-MT + a(1-\sqrt{2})]/2$$

we have

$$G(k) \geq \int_0^T g(k + (MT/2))dt = g(k + (MT/2))T = g(a(1-\sqrt{2})/2)T = a^2T/4.$$

Case 2. $\int_0^T r(t, \phi(t-h))dt > 0$. If $k = (a - MT)/2$ then we have

$$G(k) \leq \int_0^T g(k - (MT/2))dt = g(k - (MT/2))T = g((a/2) - MT)T.$$

We then need $a/2 > MT$ and $g((a/2) - MT) < -\alpha M$. A calculation yields $g((a/2) - MT) = -((a^2/4) - M^2T^2)$ which is less than $-\alpha M$ just in case the stated condition of the theorem holds.

Hence, we can find a unique

$$k \in [-MT + (a/2)(1-\sqrt{2}), (a - MT)/2] \text{ with } P\phi \in \mathcal{P}_T^0.$$

Next,

$$\|P\phi\| \leq \alpha M + g(-MT + (a/2)(1-\sqrt{2})) + \|p\| =: H \tag{3.34}$$

by (3.24), so $P : \mathcal{S} \to \mathcal{S}$.

Referring now to Krasnoselskii's theorem, we must decompose P into a sum $P = B + A$. The first impulse is to let $B = r$ and $A = g + p$. But that will not work because of the mean value. Instead, we define

$$(B\phi)(t) = r(t, \phi(t-h)) - (1/T)\int_0^T r(t, \phi(t-h))dt \tag{3.35}$$

and

$$(A\phi)(t) = g(k + \int_0^t \phi(s)ds) - (1/T)\int_0^T g(k + \int_0^t \phi(s)ds)dt + p(t). \tag{3.36}$$

By the choice of k we have

$$\int_0^T r(t, \phi(t-h))dt = -\int_0^T g(k + \int_0^t \phi(s)ds)dt. \tag{3.37}$$

By the arguments given with Theorem 3.2, A maps bounded sets into equicontinuous sets, while a simple calculation shows that B is a contraction mapping with contraction constant 2α (Note that (3.28) implies that $3\alpha \leq 1$.).

Next, if $\phi, \psi \in \mathcal{S}$ then

$$\begin{aligned} |B\phi + A\psi| &\leq |r(t, \phi(t-h))| + (1/T)\int_0^T r(t, \phi(t-h))dt| + |(1/T)\int_0^T r(t, \psi(t-h))dt| + |g(k + \int_0^t \psi(s)d \\ &\leq H + 2\alpha M \\ &\leq 3\alpha M + g(-MT + (a/2)(1-\sqrt{2})) + \|p\| \\ &\leq M \end{aligned} \tag{3.38}$$

where H is defined in (3.34) and the last inequality is obtained from (3.28).

This establishes that $B\phi + A\psi \in \mathcal{S}$ whenever $\phi, \psi \in CalS$. The conditions of Krasnoselskii's theorem are satisfied and there is a fixed point in $\mathcal{P}_T^0$.

Remark. It is a surprise that we needed $3\alpha \leq 1$. One feels that in (3.34) asking $H < M$ would suffice. But we see no way to use Krasnoselskii's theorem without the more complicated definitions of A, B and note in (3.38) that those mean values involve two entirely independent elements ϕ, ψ so we can not hope to make those mean values add up to 0.

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