THE NONLINEAR WAVE EQUATION AS A LIÉNARD EQUATION

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1. Introduction

The equation

$$u'' + f(u)u' + g(u) = 0, \quad ' = d/dt, \tag{1.1}$$

with

$$f(u) > 0 \quad \text{and} \quad ug(u) > 0 \quad \text{for} \quad u \neq 0 \tag{1.2}$$

and

$$f \text{ and } g \text{ continuous} \tag{1.3}$$

has been the subject of much interest for about sixty years and it can now be said that qualitative properties are well known. Writing (1.1) as the system

$$\begin{cases} u' = y \\ y' = -f(u)y - g(u), \end{cases} \tag{1.4}$$

the main questions were:

(A) Determine conditions under which all solutions $(u(t), y(t))$ are bounded.

(B) Determine conditions under which all solutions $(u(t), y(t)) \to (0,0)$ as $t \to \infty$.

Much of the investigation was carried out by means of Liapunov's direct method and the natural Liapunov function for (1.4) is an energy expression (which goes back to Lagrange)

$$V_1(u,y) = 2\int_0^u g(s)ds + y^2. \tag{1.5}$$

Then along a solution of (1.4) we have $V_1(u(t), y(t)) = V_1(t)$ satisfying

$$V_1'(t) = -2f(u)y^2 \leq 0. \tag{1.6}$$

Thus, for any solution $(u(t), y(t))$, both $V(t)$ and $y(t)$ are bounded. From the bound on y it follows readily that solutions can be defined for all future time.

But this Liapunov function has two deficiencies:

(C) If $G(u) = \int_0^u g(s)ds$ is bounded either for $u > 0$ or for $u < 0$, then the fact that $V_1(t)$ is bounded does not imply that $u(t)$ is bounded.

(D) Since $V_1'(t)$ is not negative definite, it is not clear that a bounded solution $(u(t), y(t))$ tends to zero as $t \to \infty$.

Problem (D) was resolved by a result of Barbashin [3] (obtained independently by LaSalle [11]) which states that if $M = \{(u, y)|V_1'(u, y) = 0\}$ and if E is the largest invariant set in M, then every bounded solution approaches E as $t \to \infty$. Here, $M = \{(0, 0)\}$.

While this is a very useful result, it is still highly advantageous to find a Liapunov function having a negative definite derivative when questions of added perturbations or delays are considered.

Both difficulties (C) and (D) were resolved by noting that the Liénard plane

$$\begin{cases} u' = z - \int_0^u f(s)ds \\ z' = -g(u) \end{cases} \tag{1.7}$$

can utilize the same general form of Liapunov function

$$V_2(u, z) = 2\int_0^u g(s)ds + z^2 \tag{1.8}$$

with derivative along a solution satisfying

$$V_2'(t) = -2g(u)\int_0^u f(s)ds \le 0. \tag{1.9}$$

Thus, by defining

$$V(u, y) = V_1 + V_2 = 4\int_0^u g(s)ds + y^2 + \left(y + \int_0^u f(s)ds\right)^2 \tag{1.10}$$

we have (along solutions of (1.4))

$$V'(t) = -2\left[g(u)\int_0^u f(s)ds + f(u)y^2\right] \tag{1.11}$$

which is negative definite. Using these ideas, we showed [4] that all solutions of (1.4) are bounded if and only if

$$\int_0^{\pm\infty} [f(u) + |g(u)|]du = \pm\infty. \tag{1.12}$$

Numerous authors have considered the wave equation (and related problems) in one or more spatial dimensions with some sort of damping. It is both interesting and useful to note that there is a "Liénard plane" for many of these equations and that there is a second natural Liapunov function which results in analysis parallel to the foregoing.

2. First order damping

The problem to be considered here is

$$u_{tt} = (a(x)g(u_x))_x - f(u)u_t, \qquad u(t,0) = u(t,1) = 0, \tag{2.1}$$

where

$$xg(x) > 0 \quad \text{if} \quad x \neq 0, \qquad a \text{ and } f \text{ are positive} \tag{2.2}$$

and

$$a,\ f,\ \text{and } g \text{ are continuous on } \mathbf{R}\,. \tag{2.3}$$

Such equations, as well as perturbed forms, are considered in ([1], [7], [16]), for example, although usually as

$$u_{tt} = u_{xx} - f(u)u_t, \qquad u(t,0) = u(t,1) = 0.$$

In the generality considered here, no claim is being made concerning existence of solutions. This study centers on forms of *a priori* bounds, a usual first step in proving existence. These results here concern boundedness and stability of solutions *so long as they exist*. Methods of proving existence are found in [6] and [16], for example.

The usual system for (2.1) is

$$\begin{cases} u_t = v \\ v_t = (a(x)g(u_x))_x - f(u)v \end{cases} \tag{2.4}$$

and it has a well-known Liapunov function for a given solution $(u(t,x), v(t,x))$ given by

$$V_1(t) = \int_0^1 \left[2\int_0^{u_x} a(x)g(s)ds + v^2\right]dx. \tag{2.5}$$

Then

$$\begin{aligned} V_1'(t) &= \int_0^1 [2a(x)g(u_x)u_{xt} + 2vv_t]dx \\ &= 2a(x)g(u_x)u_t \mid_0^1 \\ &\qquad + \int_0^1 \{-2(a(x)g(u_x))_x u_t + 2u_t[(a(x)g(u_x)_x - f(u)u_t]\}dx \end{aligned}$$

(by the induced boundary conditions $u_t(t,0) = u_t(t,1) = 0$)

$$= \int_0^1 -2f(u)u_t^2 dx$$

so that

$$V_1'(t) = -2\int_0^1 f(u)v^2 dx, \tag{2.6}$$

closely analogous to (1.6).

Problems (C) and (D) of Section 1 arise here as well. Results analogous to that of Barbashin are applied to show that bounded solutions approach certain sets where V' is zero ([2], [7], [9], [16], [17]).

But no one seems to have noticed that there is an effective "Liénard plane"

$$\begin{cases} u_t = z - \int_0^u f(s)ds \\ z_t = (a(x)g(u_x))_x \end{cases} \tag{2.7}$$

with Liapunov function of the same form

$$V_2(t) = \int_0^1 \left[2\int_0^{u_x} a(x)g(s)ds + z^2\right] dx \tag{2.8}$$

having derivative

$$\begin{aligned} V_2'(t) &= \int_0^1 \left[2a(x)g(u_x)u_{xt} + 2\left(u_t + \int_0^u f(s)ds\right)(a(x)g(u_x))_x\right] dx \\ &= 2a(x)g(u_x)u_t\,|_0^1 \\ &\quad + \int_0^1 \Big[-2(a(x)g(u_x))_x u_t + 2u_t(a(x)g(u_x))_x \\ &\quad + 2(a(x)g(u_x))_x \int_0^u f(s)ds\Big] dx \\ &= 2\int_0^1 (a(x)g(u_x))_x \int_0^u f(s)ds\,dx \end{aligned}$$

(a form analogous to (1.9) when we realize that $(a(x)g(u_x))_x$ with the stated boundary conditions is a negative operator)

$$= 2a(x)g(u_x)\int_0^u f(s)ds\,|_0^1 - 2\int_0^1 a(x)g(u_x)f(u)u_x dx$$

or

$$V_2'(t) = -2\int_0^1 a(x)g(u_x)u_x f(u)dx \le 0. \tag{2.9}$$

When we form $V(t) = V_1(t) + V_2(t)$ we obtain

$$V(t) = \int_0^1 \left[4\int_0^{u_x} a(x)g(s)ds + u_t^2 + \left(u_t + \int_0^u f(s)ds\right)^2\right] dx \tag{2.10}$$

with

$$V'(t) = -2\int_0^1 [f(u)u_t^2 + a(x)g(u_x)u_x f(u)]dx. \tag{2.11}$$

These results can be summarized as follows.

THEOREM 1. Suppose that $u(t,x)$ is a solution of (2.1) for $0 \le t < \infty$ and let (2.2) and (2.3) hold. Then for $V(t)$ defined by (2.10), $V(t)$ is nonnegative and satisfies (2.11).

Various conclusions can be drawn depending on the properties ascribed to a, f, and g; while Liapunov's direct method is well-known for such problems, it seems most worthwhile to make explicit consequences of (2.10) and (2.11) by presenting a simple stability analysis for the linear case. In the nonlinear case, Jensen's inequality is very effective in V' (see [13]).

EXAMPLE 1. Let $g(s) = s$, $f(u) = \alpha > 0$, $a(x) = 1$. Then

$$V(t) = \int_0^1 [2u_x^2 + v^2 + (v + \alpha u)^2]dx$$

so there is a $K > 0$ with

$$|u(t)|_{H^1}^2 + |v(t)|_{H^0}^2 \le V(t) \le K\big(|u(t)|_{H^1}^2 + |v(t)|_{H^0}^2\big)$$

and

$$V'(t) = -2\alpha\big(|u(t)|_{H^1}^2 + |v(t)|_{H^0}^2\big)$$

with H^1 and H^0 being the standard Sobolev norms. Incidentally, according to Pazy [14; p. 220] the spaces H^1 and H^0 are appropriate here.

We now show that the zero solution of this linearized (2.4) is uniformly asymptotically stable in $|u(t)|_{H^1}^2 + |v(t)|_{H^0}^2$. An abstract proof using dynamical system theory is found in Henry [9; pp. 82–97].

PROOF. Given $\epsilon > 0$ we must find $\delta > 0$ such that

$$|u(0)|^2_{H^1} + |v(0)|^2_{H^0} < \delta \quad \text{implies that}$$
$$|u(t)|^2_{H^1} + |v(t)|^2_{H^0} < \epsilon \quad \text{for } t > 0;$$

as the system is autonomous, there is uniform stability. Take $K\delta = \epsilon$. Then $V'(t) \leq 0$ implies that

$$|u(t)|^2_{H^1} + |v(t)|^2_{H^0} \leq V(t) \leq V(0) \leq K(|u(0)|^2_{H^1} + |v(0)|^2_{H^0}) < K\delta = \epsilon,$$

as required.

Next, for a given $L > 0$ and $\mu > 0$ we must find $T > 0$ such that $|u(0)|^2_{H^1} + |v(0)|^2_{H^0} < L$ and $t \geq T$ imply that $|u(t)|^2_{H^1} + |v(t)|^2_{H^0} < \mu$.

Pick $\eta = \mu/K$ and let $|u(0)|^2_{H^1} + |v(0)|^2_{H^0} < L$. Next, note that

(a) if there is a $t_1 > 0$ with $|u(t_1)|^2_{H^1} + |v(t_1)|^2_{H^0} < \eta$, then for $t \geq t_1$ we have $|u(t)|^2_{H^1} + |v(t)|^2_{H^0} \leq V(t) \leq V(t_1) < K\eta = \mu$;

(b) as long as $|u(t)|^2_{H^1} + |v(t)|^2_{H^0} \geq \eta$ then $V'(t) \leq -2\alpha\eta$ and so $0 \leq V(t) \leq V(0) - 2\alpha\eta t < KL - 2\alpha\eta t$, a contradiction if $t > KL/2\alpha\eta =: T$.

This completes the proof.

A wealth of results parallel to those of the example can be obtained under various assumptions on the functions. The fact that no reference need be made to invariant sets means that geneal perturbations can be considered. In the last section we perturb the equation with a delay term $u(t-1,x)c(t)$ and get a type of asymptotic stability.

3. Third order damping

Greenberg, MacCamy, and Mizel [6] consider the equation

$$\rho_0 u_{tt} = (\sigma(u_x))_x + \lambda u_{xtx}, \quad u(t,0) = u(t,1) = 0,$$

using the term λu_{xtx} as damping. This type of problem can also be treated in the "Liénard plane".

Consider the equation

$$u_{tt} = (g(u_x))_x + (h(u_x))_{xt}, \quad u(t,0) = u(t,1) = 0, \tag{3.1}$$

with

$$xg(x) > 0 \text{ if } x \neq 0, \tag{3.2}$$

$$g'(x) \text{ and } h'(x) \text{ positive and continuous on } \mathbf{R}. \tag{3.3}$$

Let

$$\begin{cases} u_t = v \\ v_t = (g(u_x))_x + (h(u_x))_{xt} \end{cases} \tag{3.4}$$

and define

$$V_1(t) = \int_0^1 \left[2\int_0^{u_x} g(s)ds + v^2 \right] dx \tag{3.5}$$

so that

$$\begin{aligned} V_1'(t) &= \int_0^1 \{2g(u_x)u_{xt} + 2u_t[(g(u_x))_x + (h(u_x))_{xt}]\}dx \\ &= 2g(u_x)u_t \mid_0^1 + \int_0^1 \{-2(g(u_x))_x u_t + 2u_t[(g(u_x))_x + (h(u_x))_{xt}]\}dx \\ &= \int_0^1 2u_t(h(u_x))_{xt} dx \\ &= 2u_t(h(u_x))_t \mid_0^1 -2\int_0^1 h(u_x)_t u_{tx} dx \end{aligned}$$

so that

$$V_1'(t) = -2\int_0^1 h'(u_x)u_{tx}^2 dx. \tag{3.6}$$

Next, let

$$\begin{cases} u_t = z + (h(u_x))_x \\ z_t = (g(u_x))_x \end{cases} \tag{3.7}$$

and define

$$V_2(t) = \int_0^1 \left[2\int_0^{u_x} g(s)ds + z^2 \right] dx \tag{3.8}$$

with

$$\begin{aligned} V_2'(t) &= \int_0^1 [2g(u_x)u_{xt} + 2zz_t]dx \\ &= 2g(u_x)u_t \mid_0^1 + \int_0^1 \{-2(g(u_x))_x u_t + 2[u_t - (h(u_x))_x](g(u_x))_x\}dx \\ &= -2\int_0^1 (h(u_x))_x (g(u_x))_x dx \end{aligned}$$

so that

$$V_2'(t) = -2\int_0^1 h'(u_x)g'(u_x)u_{xx}^2 dx. \tag{3.9}$$

Forming $V(t) = V_1(t) + V_2(t)$ we have

$$V(t) = \int_0^1 \left\{4\int_0^{u_x} g(s)ds + u_t^2 + [u_t - (h(u_x))_x]^2\right\}dx \tag{3.10}$$

with

$$V'(t) = -2\int_0^1 [h'(u_x)u_{tx}^2 + h'(u_x)g'(u_x)u_{xx}^2]dx. \tag{3.11}$$

It is noted in [6; p. 717] that if $\phi \in C^2$ on $[0,\infty)\times[0,1]$, $\phi(t,0) = \phi(t,1) = 0$ for $t \geq 0$ then

$$\begin{aligned}|\phi(t)|_{L^2} \leq |\phi(t)|_\infty \leq |\phi_x(t)|_{L^2} \leq |\phi_x(t)|_\infty \\ \leq |\phi_{xx}(t)|_{L^2} \leq |\phi_{xx}(t)|_\infty\end{aligned}$$

(all of these norms are with respect to x). Thus, if

$$h'(r) \geq c_1 > 0 \quad \text{and} \quad h'(r)g'(r) \geq c_2 > 0 \quad \text{for} \quad -\infty < r < \infty \tag{3.12}$$

then there is a $\beta > 0$ with

$$V'(t) \leq -\beta\int_0^1 [u_t^2 + u^2 + u_x^2 + u_{xx}^2]dx. \tag{3.13}$$

From (3.10) we have

$$\begin{aligned}V(t) &= \int_0^1 \left\{4\int_0^{u_x} g(s)ds + u_t^2 + [u_t - (h(u_x))_x]^2\right\}dx \\ &= \int_0^1 \left\{4\int_0^{u_x} g(s)ds + u_t^2 + [u_t - h'(u_x)u_{xx}]^2\right\}dx \\ &= \int_0^1 \left\{4\int_0^{u_x} g(s)ds + 2\left[u_t - \frac{1}{2}h'(u_x)u_{xx}\right]^2 + \frac{1}{2}h'(u_x)^2u_{xx}^2\right\}dx \\ &\geq \frac{1}{4}\int_0^1 \{u_t^2 + c_1^2u_{xx}^2\}dx \\ &\geq c_3\int_0^1 \{u^2 + u_t^2 + u_x^2 + u_{xx}^2\}dx.\end{aligned}$$

These results are summarized as follows.

THEOREM 2. Let $u(t,x)$ satisfy (3.1) on $0 \le t < \infty$. If (3.2), (3.3), and (3.12) hold, then $V(t)$ defined in (3.10) is positive definite and satisfies (3.13).

4. A three dimensional problem

Webb [16] generalizes the problem of Section 3 to three spatial dimensions for the linear case, but adds a nonlinear perturbation. He considers

$$w_{tt} - \alpha\Delta w_t - \Delta w = f(w)$$

where $\alpha > 0$ and $w(x,t) = 0$ for $x \in \partial\Omega$, $t \ge 0$ with Ω a smooth bounded domain in $\mathbf{R}^n$ for $n = 1,2,3$. Webb assumes that $f'(x) \le c_0$ for all $x \in \mathbf{R}$ with $c_0 \ge 0$, while $\limsup_{|x|\to\infty} f(x)/x \le 0$ and $f(0) = 0$.

The Liénard type transformation yields positive results for

$$u_{tt} = g_1(u_x)_x + g_2(u_y)_y + g_3(u_z)_z + h_1(u_x)_{xt} + h_2(u_y)_{yt} + h_3(u_z)_{zt},$$

but in our discussion here we restrict attention to

$$u_{tt} = \Delta u + \alpha\Delta u_t, \quad u = 0 \text{ on } \partial\Omega. \tag{4.1}$$

Define

$$\begin{cases} u_t = v \\ v_t = \Delta u + \alpha\Delta v \end{cases} \tag{4.2}$$

and

$$V_1(t) = \int_\Omega \frac{1}{2}[\nabla^2 u + v^2]dx\,dy\,dz$$

so that

$$V_1'(t) = \int_\Omega [u_x u_{xt} + u_y u_{yt} + u_z u_{zt} + u_t(\Delta u + \alpha\Delta u_t)]dw.$$

Here, $dw = dx\,dy\,dz$; subsequently, dS is the differential of surface on $\partial\Omega$.

Using the divergence theorem and taking into account that $u = 0$ on $\partial\Omega$, we have

$$V_1'(t) = -\alpha\int_\Omega (u_{tx}^2 + u_{ty}^2 + u_{tz}^2)dw. \tag{4.3}$$

More detail will be given for the more complicated case of V_2.

Next, use the Liénard transformation

$$\begin{cases} u_t = p + \alpha\Delta u \\ p_t = \Delta u \end{cases} \tag{4.4}$$

with

$$V_2(t) = \int_\Omega \frac{1}{2}[\nabla^2 u + p^2]dw \tag{4.5}$$

so that

$$V_2'(t) = \int_\Omega [u_x u_{xt} + u_y u_{yt} + u_z u_{zt} + p(u_{xx} + u_{yy} + u_{zz})]dw.$$

Now the divergence theorem is

$$\int_\Omega \nabla \cdot F\, dw = \int_{\partial\Omega} n \cdot F\, dS$$

where $dw = dx\, dy\, dz$, n is the normal, and dS is the surface differential. If we take $F = \phi\nabla\psi$ then $\nabla \cdot F = \phi\nabla \cdot \nabla\psi + \nabla\phi \cdot \nabla\psi$. Letting $u = \psi$ and $u_t = \phi$ we have $\nabla \cdot F = u_t\nabla \cdot \nabla u + \nabla u_t \cdot \nabla u = u_t\Delta u + (u_{tx}u_x + u_{ty}u_y + u_{tz}u_z)$. Thus,

$$\int_\Omega u_t\Delta u\, dw + \int_\Omega (u_{tx}u_x + u_{ty}u_y + u_{tz}u_z)dw = \int_{\partial\Omega} n \cdot \phi\nabla\psi\, dS = 0$$

and so

$$\int_\Omega (u_{tx}u_x + u_{ty}u_y + u_{tz}u_z)dw = \int_\Omega -u_t(u_{xx} + u_{yy} + u_{zz})dw.$$

Hence,

$$V_2'(t) = \int_\Omega [-u_t\Delta u + (u_t - \alpha\Delta u)\Delta u]dw$$

so that

$$V_2'(t) = -\alpha \int_\Omega (\Delta u)^2 dx\, dy\, dz. \tag{4.6}$$

Taking $V(t) = V_1(t) + V_2(t)$ we get

$$V(t) = \int_\Omega [\nabla^2 u + u_t^2 + (u_t - \alpha\Delta u)^2]dx\, dy\, dz \tag{4.7}$$

with

$$V'(t) = -\alpha \int_\Omega [u_{tx}^2 + u_{ty}^2 + u_{tz}^2 + (\Delta u)^2]dx\, dy\, dz. \tag{4.8}$$

Under the conditions here, it is shown in Simpson and Spector [15; p. 26] that there is a $\gamma > 0$ with

$$\gamma \int_\Omega (u_x^2 + u_y^2 + u_z^2) dw \le \int_\Omega (\Delta u)^2 dw.$$

Also, Hale [7; p. 136] notes that if $\phi \in H_0^1(\Omega)$, then $|\phi|_{L^2}^2 \le |\nabla \phi|_{L^2}^2 / \lambda_1$ where λ_1 is the first eigenvalue of $-\Delta$ on H_0^1. Apply this to $\phi = u_t$ and to $\phi = u$. All of this shows that there is a $\beta > 0$ with

$$V'(t) \le -\beta \int_\Omega [u^2 + u_t^2 + u_x^2 + u_y^2 + u_z^2] dx\, dy\, dz. \tag{4.9}$$

These results are summarized as follows.

THEOREM 3. Let u satisfy (4.1) for $0 \le t < \infty$ and let $\alpha > 0$. Then $V(t)$ defined by (4.7) is nonnegative and $V'(t)$ satisfies (4.9).

5. A vector equation

Kato [10] considers a forced version of the vector equation

$$u'' + f(u)u' + g(u) = 0$$

and supposes that there is a vector function $F(u)$ with $f(u) = \text{grad } F(u)$, and a scalar function $G(u)$ for which $\text{grad } G(u) = g(u)$. Here, $f(u)$ is an $n \times n$ matrix, u and g are n-vectors. Kato assumes variants of $u^T g(u) > 0$, $g^T(u)F(u) > 0$, $u^T f(v)u > 0$ for $u \ne 0$ and all v. As he seeks boundedness, he usually asks that these conditions hold for $|u| \ge a$, $a \ge 0$. The equation may be written as

$$\begin{cases} u' = v \\ v' = -f(u)v - g(u) \end{cases}$$

or as the Liénard system

$$\begin{cases} u' = z - F(u) \\ z' = -g(u) \end{cases}$$

and utilize the natural (energy) Liapunov functions

$$V_1(u, v) = G(u) + \frac{1}{2} v^T v$$

or

$$V_2(u, z) = G(u) + \frac{1}{2} z^T z.$$

Parallel to this, we consider

$$u_{tt} = -f(u)u_t + g(u_x)_x \tag{5.1}$$

where $u = u(t,x)$, $u = (u_1, ..., u_n)$, $u(t,0) = u(t,1) = 0$, t and x are scalars, there are F and G with

$$f(u) = \text{grad } F(u), \quad g(u) = \text{grad } G(u), \tag{5.2}$$

$$\begin{cases} u^T g(u) > 0, & g^T(u)F(u) > 0, \quad g^T(u)f(v)u > 0 \\ u^T f(v)u > 0, & \text{if } u \neq 0. \end{cases} \tag{5.3}$$

Write (5.1) as

$$\begin{cases} u_t = v \\ v_t = -f(u)v + g(u_x)_x \end{cases} \tag{5.4}$$

and define

$$V_1(t) = \int_0^1 \left[G(u_x) + \frac{1}{2}v^T v\right] dx \tag{5.5}$$

so that

$$\begin{aligned} V_1'(t) &= \int_0^1 [g^T(u_x)u_{xt} + v^T v_t]dx \\ &= g^T(u_x)u_t \mid_0^1 + \int_0^1 [-g^T(u_x)_x u_t + v^T(-f(u)v + g(u_x)_x)]dx \end{aligned}$$

and therefore

$$V_1'(t) = -\int_0^1 v^T f(u)v \, dx \leq 0. \tag{5.6}$$

Next, write (5.1) as

$$\begin{cases} u_t = z - F(u) \\ z_t = g(u_x)_x \end{cases} \tag{5.7}$$

and define

$$V_2(t) = \int_0^1 \left[G(u_x) + \frac{1}{2}z^T z\right] dx$$

with

$$\begin{aligned} V_2'(t) &= \int_0^1 [g^T(u_x)u_{xt} + z^T z_t]dx \\ &= g^T(u_x)u_t \mid_0^1 + \int_0^1 [-g^T(u_x)_x u_t + z^T g(u_x)_x]dx \\ &= \int_0^1 [-g^T(u_x)_x u_t + g^T(u_x)_x(u_t + F(u))]dx \\ &= \int_0^1 g^T(u_x)_x F(u)dx = g^T(u_x)F(u) \mid_0^1 - \int_0^1 g^T(u_x)f(u)u_x dx \end{aligned}$$

or

$$V_2'(t) = -\int_0^1 g^T(u_x)f(u)u_x dx. \tag{5.9}$$

Taking $V(t) = V_1(t) + V_2(t)$ we have

$$V(t) = \int_0^1 \left[2G(u_x) + \frac{1}{2}u_t^T u_t + \frac{1}{2}(u_t + F(u))^T(u_t + F(u))\right] dx \tag{5.10}$$

and

$$V'(t) = -\int_0^1 [u_t^T f(u)u_t + g^T(u_x)f(u)u_x]dx. \tag{5.11}$$

These results are summarized as follows.

THEOREM 4. Let $u(t, x)$ satisfy (5.1) for $0 \le t < \infty$ and let (5.3) hold. Then $V(t)$ defined by (5.10) is nonnegative and satisfies (5.11).

6. Applications

Boundedness of solutions and asymptotic stability of an unperturbed equation can frequently be proved using a Liapunov function whose derivative is negative semi-definite. But when arbitrary (bounded) perturbations or delays are added, then the same Liapunov function will generally not yield boundedness. The book by Hale [7] shows numerous examples of the type displayed here using only the V_1; he can even deal with large perturbations so long as they are of a very special type. But on pp. 132–3 he notes that when a periodic perturbation is added, then changes in the Liapunov function must be made. He follows the changes made by Ghidaghlia and Témam [5], Babin and Vishik [2], Haraux [8], and Lopes and Ceron [12] which are largely exercises in desperation that work to a degree for quadratic forms. The Liénard transformation is a formal solution to achieving a negative definite derivative as well as providing Liapunov functions which are radially unbounded.

Moreover, the form of $V_1 + V_2$ has proved to be fundamental in that for the ODE case, the zero solution is globally asymptotically stable if and only if $V_1 + V_2$ is radially unbounded. For these PDE's, owing to the far more complex spaces, it remains to be seen whether or not the $V_1 + V_2$ will prove to be so fundamental. That is a question which will require much time, space, and effort to resolve.

The examples presented here do not show the power of the foregoing results; instead, they show only the types of consequences.

EXAMPLE 2. Every solution $u(t, x)$ of

$$u_{tt} = (u_x e^{-u_x^2})_x - u_t, \qquad u(t, 0) = u(t, 1) = 0, \tag{6.1}$$

which exists on $0 \le t < \infty$ is bounded in the sense that $\int_0^1 [u^2(t,x)+u_t^2(t,x)]dx$ is bounded.

PROOF. The equation is of the form of (2.1) with $a(x) = 1$, $g(x) = xe^{-x^2}$, and $f(x) = 1$. Using (2.10) we have

$$\int_0^1 [u_t^2 + (u_t + u)^2]dx \le V(t) \le V(0)$$

since $V'(t) \le 0$. The required inequality is now immediate.

Note that boundedness from V_1 does not seem to follow.

EXAMPLE 3. Consider the equation

$$u_{tt} = g(u_x)_x - f(u)u_t + e(t), \quad u(t,0) = u(t,1) = 0 \tag{6.2}$$

with $xg(x) > 0$ if $x \ne 0$, $xg(x) \ge \alpha_1 x^2 - \beta_1$, $\alpha_2 x^2 - \beta_2 \le \int_0^x g(s)ds \le \alpha_3 x^2 + \beta_3$, $1 \le f(x) \le \beta_4$ for all x and some positive constants α_i and β_i, $e(t)$ is bounded and continuous. Then there is an $M > 0$ such that any solution of (6.2) on $[0,\infty)$ satisfies

$$|u(t,x)| + \int_0^1 [u^2(t,x) + u_t^2(t,x) + u_x^2(t,x)]dx \le M$$

for all large t.

PROOF. From (2.10) we have

$$V(t) = \int_0^1 \left[4\int_0^{u_x} g(s)ds + u_t^2 + \left(u_t + \int_0^u f(s)ds\right)^2\right] dx$$

so that

$$\begin{aligned}
V'(t) &= \int_0^1 \left[4g(u_x)u_{xt} + 2u_t u_{tt} + 2\left(u_t + \int_0^u f(s)ds\right)(u_{tt} + f(u)u_t)\right] dx \\
&= \int_0^1 \Big\{ -4g(u_x)_x u_t + 2u_t[g(u_x)_x - f(u)u_t + e(t)] \\
&\quad + 2\left(u_t + \int_0^u f(s)ds\right)[g(u_x)_x - f(u)u_t + e(t) + f(u)u_t]\Big\} dx \\
&= \int_0^1 \Big\{ -2g(u_x)_x u_t - 2f(u)u_t^2 + 2u_t e(t) + 2g(u_x)_x u_t \\
&\quad + 2u_t e(t) + 2g(u_x)_x \int_0^u f(s)ds + 2e(t)\int_0^u f(s)ds\Big\} dx \\
&= \int_0^1 \Big\{ -2f(u)u_t^2 + 4u_t e(t) - 2f(u)g(u_x)u_x + 2e(t)\int_0^u f(s)ds\Big\} dx \\
&\le \int_0^1 \left\{-2u_t^2 + 4|u_t||e(t)| - 2g(u_x)u_x + 2|e(t)|\left|\int_0^u f(s)ds\right|\right\} dx \\
&\le \int_0^1 \left\{-2u_t^2 + 4|u_t||e(t)| - 2\alpha_\infty u_x^2 + 2\beta_\infty + 2|e(t)|\left|\int_0^u f(s)ds\right]\right\} dx.
\end{aligned}$$

Now there exist M_1, M_2 with $4|u_t||e(t)| \leq u_t^2 + M_1$ and $|e(t)|\left|\int_0^u f(s)ds\right| \leq (\alpha_1/2)u^2 + M_2$, while $\int_0^1 u_x^2 dx \geq \pi^2 \int_0^1 u^2 dx$. Thus, there are positive constants γ and $\overline{M}$ with

$$V'(t) \leq -\gamma \int_0^1 [u^2 + u_x^2 + u_t^2]dx + \overline{M}.$$

Now $V(t) \leq \int_0^1 \left[4(\alpha_3 u_x^2 + \beta_3) + 3u_t^2 + 2\left(\int_0^u f(s)ds\right)^2\right] dx \leq \int_0^1 [4\alpha_3 u_x^2 + 3u_t^2 + 2\beta_4^2 u^2]dx$ $+4\beta_3$ and so there are positive constants c_1 and c_2 with $V'(t) \leq -c_1 V + c_2$. The conclusion now follows from standard arguments.

EXAMPLE 4. Consider the equation

$$\begin{gathered} u_{tt} = (u_x + u_x^3)_x - \alpha u_t + c(t)u(t-1, x), \\ u(t,0) = u(t,1) = 0, \end{gathered} \tag{6.3}$$

where $c(t)$ is continuous on $[0, \infty)$, $|c(t)| \leq 1 < \alpha$, α constant. Then each solution defined on $[0, \infty)$ satisfies

$$|u(t,x)| + \int_0^1 [u^2 + u_x^2 + u_x^4 + u_t^2]dx \to 0 \quad \text{as} \quad t \to \infty.$$

PROOF. There is, of course, an implied initial condition on $[-1, 0]$. We add the usual term in such problems to the V defined in (2.10) and have

$$V(t) = \int_0^1 \left[4\int_0^{u_x} g(s)ds + u_t^2 + (u_t + \alpha u)^2 + \alpha \int_{t-1}^t |c(s+1)|u_x^2(s,x)ds\right] dx$$

where $g(x) = x + x^3$. Then an integration by parts yields

$$\begin{aligned} V'(t) &= \int_0^1 [-4g(u_x)_x u_t + 2u_t u_{tt} + 2(u_t + \alpha u)(u_{tt} + \alpha u_t) \\ &\qquad + \alpha|c(t+1)|u_x^2(t,x) - \alpha|c(t)|u_x^2(t-1,x)\big]\,dx \\ &= \int_0^1 \{-4g(u_x)_x u_t + 2u_t[g(u_x)_x - \alpha u_t + c(t)u(t-1,x)] \\ &\qquad + 2(u_t + \alpha u)[g(u_x)_x + c(t)u(t-1,x)] \\ &\qquad + \alpha|c(t+1)|u_x^2(t,x) - \alpha|c(t)|u_x^2(t-1,x)\big\}\,dx \\ &= \int_0^1 \big\{-2g(u_x)_x u_t - 2\alpha u_t^2 + 2c(t)u_t u(t-1,x) \\ &\qquad + 2u_t g(u_x)_x + 2u_t c(t)u(t-1,x) + 2\alpha u g(u_x)_x \\ &\qquad + 2\alpha c(t)uu(t-1,x) + \alpha|c(t+1)|u_x^2(t,x) - \alpha|c(t)|u_x^2(t-1,x)\big\}\,dx \\ &= \int_0^1 \big\{-2\alpha u_t^2 + 4c(t)u_t u(t-1,x) + 2\alpha c(t)uu(t-1,x) \\ &\qquad - 2\alpha g(u_x)u_x + \alpha|c(t+1)|u_x^2(t,x) - \alpha|c(t)|u_x^2(t-1,x)\big\}\,dx \\ &\le \int_0^1 \big\{-2\alpha u_t^2 + 2|c(t)|[u_t^2 + u^2(t-1,x)] \\ &\qquad + \alpha|c(t)|[u^2 + u^2(t-1,x)] - 2\alpha u_x^2 - 2\alpha u_x^4 \\ &\qquad + \alpha|c(t+1)|u_x^2(t,x) - \alpha|c(t)|u_x^2(t-1,x)\big\}\,dx \\ &\le \int_0^1 \big\{-2(\alpha - |c(t)|)u_t^2 + \alpha\big[-2 + (|c(t)|/\pi^2) + |c(t+1)|\big]u_x^2 \\ &\qquad - 2\alpha u_x^4 + (-\alpha + [(\alpha+2)/\pi^2]/|c(t)|u_x^2(t-1,x)\big\}\,dx \\ &\le \int_0^1 \big\{2(-\alpha+1)u_t^2 - \alpha[1 - (1/\pi^2)]u_x^2 - 2\alpha u_x^4 \\ &\qquad - (1/2)|c(t)|u_x^2(t-1,x)\big\}\,dx. \end{aligned}$$

If we integrate V' from 0 to ∞, we see that $\int_0^1 \int_{t-1}^t u_x^2(s,x)ds\,dx \to 0$ as $t \to \infty$. Since $V'(t) \le 0$, $V(t) \to c \ge 0$. If $c > 0$, then for large t, $\int_0^1 [(1/2)u_x^2 + (1/4)u_x^4 + u_t^2]dx \ge c/2$ and so $V'(t) \le -\gamma < 0$ for large t. This means that $c = 0$ and the conclusion follows from a Sobolev inequality.

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