AN EXISTENCE THEOREM FOR A NEUTRAL EQUATION

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Abstract. Classical theory of neutral equations treats problems of the form $(d/dt)D(t, x_t) = f(t, x_t)$. In this paper we prove an existence theorem for

$$x'(t) = f(t, x(t), x'(t - h(t))) + g(t, x(t), x(t - h(t))).$$

Parallel results can be obtained in the same way for more general delays. The proof rests on a fixed point theorem of Krasnoselskii based on a map of the differential equation itself, not on an integral equation.

1. Introduction. A neutral functional differential equation is one in which the derivatives of the past history are involved, as well as those of the present state of the system. Thus, it is reasonable to expect a neutral equation to have a form such as

$$x'(t) = f(t, x(t), x'(t - h(t))) + g(t, x(t), x(t - h(t))) \tag{1}$$

where $0 \leq h(t) \leq h_0$, for some $h_0 > 0$. But a survey of the literature shows that what is actually studied is

$$\frac{d}{dt}D(t, x_t) = f(t, x_t) \tag{2}$$

where $x_t(s) = x(t + s)$, $s \leq 0$. And there is technical necessity for this. If D and f are continuous and D is Lipschitz in the second argument, then an integration yields the sum of a contraction and compact maps. Darbo's fixed point theorem applies and the fixed point solves the existence problem.

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By contrast, if in (1) we have f and g continuous and f Lipschitz in the second and third arguments, can we say that

$$\int f(t,x(t),x'(t-h(t)))dt =: F(t,x(t),x(t-h(t)))$$

where F is Lipschitz in the second and third arguments? In general, we do not know the answer, but our ignorance led to a nice solution to the problem.

Instead of integrating (1) to obtain an integral equation as a mapping for fixed points, use the right-hand-side of (1) itself as the mapping. It turns out that this is then the sum of a contraction and compact maps. Darbo's theorem seems to have problems here, but a slight variant of Krasnoselskii's fixed point theorem yields a solution.

2. The literature. Neutral equations have been studied for a long time, and with good reason. On the intuitive level, every parent, every gardener, and every stock broker has observed growth spurts; present growth rate is closely tied to recent growth rate. And this is the very essence of neutral equations.

Gopalsamy and Zhang ([2],[3]) and Kuang ([5],[6]) devote much space to population problems heuristically modeled as neutral equations. While those models begin as

$$x'(t) = x(t)[1-x(t-h)-x'(t-h)], \tag{3}$$

for example, it seems crucial that they are subsequently mapped into a form of (2), about which much theory exists. The form of (2) dictates that $x'(t-h)$ occurs linearly and this can force inappropriate assumptions out of technical necessity.

On the other hand, starting from first principles of physics, Driver [1], studies a two-body problem in terms of neutral equations. While his equation looks at first more like (1) than (2), it should be noted that it is linear in $x'(t-h)$.

The basic implications of the form of (2) are found with substantial discussions in at least two places. Hale [4] begins a discussion on p. 275–278, with additional remarks related to p. 49. A much fuller discussion is found in Lakshmikantham, Zhang, Wen [7] in

Chapters 1, 3, 5, and 6. In those discussions (2) is considered with D atomic in a sense they describe. The requirements imply four conditions which (1) avoids:

(i) The neutral equation must be linear in $x'(t-h)$.

(ii) If x multiplies $x'(t-h)$, the argument of x must also be $t-h$.

(iii) The equation must actually depend on the delay.

(iv) The zero function must be part of the mapping set for Darbo's theorem.

We avoid all of these by working directly with (1) without conversion to an integral equation. We have settled on the simple form (1) for clarity of exposition. The interested reader can carry out the arguments for more general delays.

3. Existence. We consider

$$x'(t) = f(t, x(t), x'(t - h(t))) + g(t, x(t), x(t - h(t))) \tag{1}$$

where f, g and h are continuous, $0 \leq h(t) \leq h_0$, $h_0 > 0$, f and $g : [0, \infty) \times R \times R \to R$.

For a given continuous initial function, (1) can have discontinuous solutions. We are looking for smooth solutions and that will require a function $\eta : [-h_0, 0] \to R$ with η' continuous and the derivative of η from the left at 0 satisfying

$$\eta'(0) = f(0, \eta(0), \eta'(-h(0)) + g(0, \eta(0), \eta(-h(0))). \tag{4}$$

A solution will be a function $x(t, \eta)$ on an interval $[-h_0, r)$, $r > 0$, with $x(t, \eta) = \eta(t)$ on $[-h_0, 0]$, while x satisfies (1) on $[0, r)$. With this construction, we can continue the solution on $[r, r_1]$, some $r_1 > r$, if certain conditions are met.

Consider the complete metric space $(\mathcal{S}, \rho)$ of continuous functions $\phi : [-h_0, r] \to R$ with $\phi(t) = \eta'(t)$ on $[-h_0, 0]$, and $\rho(\phi, \psi) = \|\phi - \psi\| = \sup_{0 \leq t \leq r} |\phi(t) - \psi(t)|$.

For each $\phi \in \mathcal{S}$, define

$$\Phi(t) = \begin{cases} \eta(t) & \text{if } -h_0 \leq t \leq 0 \\ \eta(0) + \displaystyle\int_0^t \phi(s)ds & \text{if } 0 \leq t \leq r. \end{cases} \tag{5}$$

Thus,

$$\Phi'(t) = \phi(t) \text{ on } [0, r]$$

and Φ' is continuous.

We now suppose that there is an $a > 0$ so that if $\mathcal{S}^*$ is the subset of $\mathcal{S}$ with

$$\mathcal{S}^* = \{\phi \in \mathcal{S} \,\big|\, |\phi(t) - \eta'(0)| \leq a\}$$

then there is an $\alpha > 0$ and $\beta < 1/2$ such that for $\phi, \psi \in \mathcal{S}^*$ and A, B defined by

$$(A\phi)(t) = g(t, \Phi(t), \Phi(t - h(t))) \tag{6}$$

and

$$(B\phi)(t) = f(t, \Phi(t), \phi(t - h(t))) \tag{7}$$

then

$$|(B\phi)(t) - (B\psi)(t)| \leq \alpha|\Phi(t) - \Psi(t)| + \beta|\phi(t - h(t)) - \psi(t - h(t))|. \tag{8}$$

THEOREM. Let η satisfy (4) and suppose that (8) holds. Then there is an $r > 0$ and a smooth solution $x(t, \eta)$ of (1) on $[0, r)$.

PROOF. First, for the $a > 0$ we can find $M(a) > 0$ so that $\phi \in \mathcal{S}^*$ implies that

$$|(B\phi)(t)| + |(A\phi)(t)| \leq M(a). \tag{9}$$

Moreover, $A\mathcal{S}^*$ is equicontinuous.

Next, for fixed $a > 0$ we can find $r > 0$ so that $\phi \in \mathcal{S}^*$ and $0 \leq t \leq r$ implies that

$$|f(t, \Phi(t), \phi(t - h(t))) - f(0, \eta(0), \eta'(-h(0)))| \leq \frac{a}{2}\,. \tag{10}$$

To see this, we have

$$\begin{aligned} |f(t, \Phi(t), &\phi(t - h(t))) - f(0, \eta(0), \eta'(-h(0)))| \\ &\leq \alpha|\Phi(t) - \eta(0)| + \beta|\phi(t - h(t)) - \eta'(-h(0))| \\ &\leq \alpha t|\phi(\xi)| + \beta|\phi(t - h(t)) - \eta'(-h(0))| \end{aligned}$$

where $0 < \xi < t$.

We first make r so small, say $r = r^*$, that $|\phi(t-h(t)) - \eta'(-h(0))| \leq a$ when $0 \leq t \leq r^*$. We show this by considering two cases.

a. If $h(0) = 0$, then $0 \leq t - h(t) \leq t$ so $\beta|\phi(t-h(t)) - \eta'(-h(0))| = \beta|\phi(t-h(t)) - \eta'(0)| \leq \beta a < a/2$ by definition of $\mathcal{S}^*$ since $\beta < 1/2$.

b. If $h(0) > 0$, then there is an $r_1 > 0$ so that $t - h(t) \leq 0$ for $0 \leq t \leq r_1$. Thus, on $[0, r_1]$, $\phi(t-h(t)) = \eta'(t-h(t))$ and by continuity of η' we have an $r_2 > 0$ with $|\eta'(t-h(t)) - \eta'(0)| < a$ if $0 \leq t \leq r_2$. We take $0 < r^* \leq \min[r_1, r_2]$.

As $\beta < 1/2$, there is a number $q < a/2$ with $\beta|\phi(t-h(t)) - \eta'(-h(0))| \leq q$. Next, since ϕ is bounded, we now choose $0 < r < r^*$ so that $0 \leq t \leq r$ implies that $\alpha t|\phi(\xi)| + q \leq a/2$, proving (10).

Next, we can make r so small that $\phi \in \mathcal{S}^*, 0 \leq t \leq r$ implies that

$$|g(t, \Phi(t), \Phi(t-h(t))) - g(0, \eta(0), \eta(-h(0)))| \leq \frac{a}{2}\,. \tag{11}$$

To see this, as g is uniformly continuous on any bounded set, if $0 \leq t - h(t)$ then

$$\begin{aligned}
&|t-0| + |\Phi(t) - \eta(0)| + |\Phi(t-h(t)) - \eta(-h(0))| \\
&\quad \leq t + t|\phi(\xi_1)| + (t-h(t))|\phi(\xi_2)| \\
&\quad \leq r\big[1 + |\phi(\xi_1)| + |\phi(\xi_2)|\big]
\end{aligned}$$

which can be made arbitrarily small. The case of $t - h(t) < 0$ follows from continuity of η. This proves (11).

Finally, from (8) we see that there is a $\gamma < 1$ so that for r sufficiently small then $\phi, \psi \in \mathcal{S}^*$ implies that

$$\|B\phi - B\psi\| \leq \gamma\|\phi - \psi\|. \tag{12}$$

We now follow the proof of Krasnoselskii's theorem [8:p.32] and show that $A + B$ has a fixed point in $\mathcal{S}^*$.

First, we show that $\phi, \psi \in \mathcal{S}^* \Rightarrow A\psi + B\phi \in \mathcal{S}^*$. To see this, we first note that

$$(A\psi)(0) + (B\psi)(0) = g(0, \eta(0), \eta(-h(0)) + f(0, \eta(0), \eta'(-h(0))) = \eta'(0). \tag{13}$$

Next, (10), (11), and (13) show that

$$|(A\psi)(t) + (B\phi)(t) - \eta'(0)| \leq \frac{a}{2} + \frac{a}{2} = a.$$

These two results show that for fixed $\psi \in \mathcal{S}^*$ then the map $\phi \in \mathcal{S}^* \Rightarrow \phi \to A\psi + B\phi \in \mathcal{S}^*$; moreover, by (12) this map is a contraction and so has a unique fixed point in $\mathcal{S}^*$: that is, for each fixed $\psi \in \mathcal{S}^*$ there is a unique $\phi \in \mathcal{S}^*$ with

$$\phi = B\phi + A\psi. \tag{14}$$

As Smart [8; p. 32] shows, $(I - B)^{-1}$ is continuous and so

$$\phi = (I - B)^{-1} A\psi \in \mathcal{S}^*$$

for each $\psi \in \mathcal{S}^*$. The map $(I - B)^{-1}A$ is continuous, while $\mathcal{S}^*$ is convex. We noted earlier that $A\mathcal{S}^*$ is equicontinuous. By continuity of $(I - B)^{-1}$, $(I - B)^{-1}AS^*$ is equicontinuous. By Schauder's second theorem $(I - B)^{-1}A$ has a fixed point $\psi \in \mathcal{S}^*$ or

$$\psi = (I - B)^{-1} A\psi$$

so that

$$\psi = B\psi + A\psi$$

and this is the required solution of (1).

References

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