Fixed Point Theory,
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FRACTIONAL EQUATIONS AND A THEOREM OF BROUWER-SCHAUDER TYPE

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Abstract. **Note.** There is an error in the published form of this paper and a correction is to appear. In the published form we asked that M be a closed, convex, bounded, nonempty set. The published form is correct in case we are working on a finite interval $0 \leq t \leq T < \infty$. But when the interval is $[0, \infty)$ then M must be a closed ball. The error occurs in the middle of the proof of Theorem 4.1 where we stated that M is closed in the weighted norm. That may not true unless M is a ball.

Brouwer's fixed point theorem states that a continuous mapping of a closed, bounded, convex, nonempty set $M \subset R^n$ into itself has a fixed point. Schauder's theorem states that a continuous mapping of a closed, convex, nonempty set M in a Banach space has a fixed point, provided that M is mapped into a compact subset of itself. In this brief note we point out that for a large class of differential equations which are transformed into integral equations defining the mapping, then that last compactness condition can be dropped, provided that M is bounded in the supremum norm. The set M is usually composed of continuous functions $\phi : [0, \infty) \to \Re$ and it can be a substantial task to prove compactness, sometimes requiring draconian conditions such as all the functions in M having the same limit at ∞. In effect, then, we reduce the conditions of Schauder's theorm (in function spaces with domains on an infinite interval) to the conditions of the far simpler Brouwer's theorem in R^n for this class of problems.

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1. Introduction: Automatic Compactness

Fixed point theory is now one of the major tools of the applied mathematician and this paper offers a significant simplification for a large class of problems of current interest. Investigators commonly study integral equations by means of Schauder's and, more generally, Krasnoselskii's fixed point theorem which contains Schauder's theorem. In these results one often sees an integral of the form

$$\int_0^t R(t-s)u(s,x(s))ds$$

which is to be compact on certain closed, bounded, convex, and nonempty sets, say M, of functions $\phi : [0,T] \to \Re$ where T can be finite or, more often, $T = \infty$ and we then more properly write $\phi : [0,\infty) \to \Re$. Fulfilling that condition has offered a significant challenge.

This note is motivated by the fact that these applications may be far simpler than they appear. In some of the most general and important cases the compactness is automatic. In fact, the conditions of Schauder's theorem coincide with those of Brouwer's fixed point theorem for R^n: If $P : M \to M$ is continuous, while $u(t,\phi(t))$ is bounded for ϕ in M, then P has a fixed point in M. The point is that **in the proper space** P is trivially compact on M even when M consists of functions on $[0,\infty)$ and it need not even be mentioned. The investigator working in applications still has the (nontrivial) task of locating M and proving continuity of the map, exactly as must be done for Brouwer's theorem. To the point of continuity, Barosso [1] and Garcia-Falset and Latrach [8], together with several colleagues, have attacked the problem from a different direction by weakening the continuity assumption in Schauder's part of Krasnoselskii's theorem. Their focus is continuity, while ours is compactness. Statements of all of these fixed point theorems are found in Smart [13].

2. The Motivating Problem

A vast collection of real-world problems is drawn from fractional differential equations of Caputo type

$$^cD^q x(t) = u(t,x(t)), \quad 0 < q < 1, \quad x(0) \in \Re \tag{1}$$

where u is continuous on $[0,T] \times \Re$ and $T \le \infty$. The Caputo fractional derivative of order q of a function x is defined to be [10, p. 12]

$$^cD^q x(t) = \frac{1}{\Gamma(1-q)} \int_{t_0}^t (t-s)^{-q} x'(s)ds. \tag{2}$$

When $u(t, x)$ is continuous then (1) is immediately inverted as the very familiar integral equation ([10, p. 54], [9, pp. 78, 86, 103])

$$x(t) = x(0) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} u(s, x(s)) ds \tag{3}$$

where Γ is the gamma function. It is natural to use (3) to define a fixed point mapping, but the singular kernel

$$\frac{t^{q-1}}{\Gamma(q)}$$

has an infinite integral which prevents us from getting a compact map of the type we would like. However, that kernel is completely monotone and by using two elementary devices we can get exactly what we want. The process was introduced in [3] and is repeated here for ready reference and for a later generalization.

Divide and multiply by $J > 0$. Then, subtract and add $x(s)$ so that we can write (3) as

$$\begin{aligned} x(t) &= x(0) \\ &+ \frac{J}{\Gamma(q)} \int_0^t (t-s)^{q-1} \left[-x(s) + x(s) + \frac{u(s, x(s))}{J} \right] ds \\ &= x(0) - \frac{J}{\Gamma(q)} \int_0^t (t-s)^{q-1} x(s) ds \\ &+ \frac{J}{\Gamma(q)} \int_0^t (t-s)^{q-1} \left[x(s) + \frac{u(s, x(s))}{J} \right] ds. \end{aligned} \tag{5}$$

Write the linear part as

$$z(t) = x(0) - \int_0^t \frac{J}{\Gamma(q)} (t-s)^{q-1} z(s) ds$$

and define the kernel as

$$C(t) = \frac{J t^{q-1}}{\Gamma(q)} \tag{6}$$

with resolvent equation

$$R(t) = C(t) - \int_0^t C(t-s) R(s) ds$$

so that by a variation of parameters formula

$$z(t) = x(0) \left[1 - \int_0^t R(s) ds \right]. \tag{7}$$

These equations are discussed in detail in [3], taking the main properties from [12, pp. 193-222], where it is shown that R satisfies

$$0 < R(t) \le \frac{J\, t^{q-1}}{\Gamma(q)}, \quad \int_0^\infty R(s) ds = 1 \tag{8}$$

so that (5) can be written as

$$x(t) = z(t) + \int_0^t R(t-s)\left[x(s) + \frac{u(s,x(s))}{J}\right]ds \tag{9}$$

where $z(t) \to 0$ as $t \to \infty$.

Next, we define a Banach space of bounded continuous functions $\phi : [0,T] \to \Re$ with the supremum norm, all denoted by

$$(BC, \|\cdot\|).$$

Then select a closed, bounded, convex, nonempty set M in BC so that when P is defined by $\phi \in M$ implies

$$(P\phi)(t) = z(t) + \int_0^t R(t-s)\left[\phi(s) + \frac{u(s,\phi(s))}{J}\right]ds \tag{10}$$

then P maps M into M. We then apply Schauder's fixed point theorem and obtain a fixed point which is a solution of (9) and it is a bounded and continuous function on $[0,T]$. Our main interest, of course, is the case where $T = \infty$.

We could continue and define a great many other kinds of mappings in the same way and obtain solutions with pre-specified limits at ∞, asymptotically periodic solutions, and solutions in $L^p[0,\infty)$, as may be seen in [4], [5], and [6], for example.

Equation (9) is the prototype and the reader is reminded that it covers a vast array of real-world problems, including many partial differential equations. But the set of problems covered by our work here goes far beyond (9) and we will briefly describe them. Explicitly, we show that the steps from (3) to (9) can also be applied to equations with an ordinary derivative and we can treat functional differential equations as the same kind of fixed point problems with the added property that R is a simple known exponential function so that the mapping of M into M is much easier to define.

3. The working theorem

In this section we will display a theorem which we propose as the working model that an investigator would use in studing qualitative properties of solutions of fractional and other types of differential equations. It is completely parallel to Brouwer's fixed point theorem requiring a closed, convex, bounded, nonempty set M and a continuous mapping patterened after (10) of M into M.

Our work will cover all scalar fractional differential equations of Caputo type when $u(t,\phi(t))$ is continuous and is bounded when $\phi \in M$, but it will also cover many equations not of this type. Thus, in this section we will deal with the Banach space $(BC, \|\cdot\|)$ of bounded continuous functions $\phi : I \to \Re$ with the supremum norm in which we have found a set M which will be described in two ways. First, if $I = [0, T]$ with $T < \infty$, then M is a closed, convex, bounded, nonempty set. But if $I = [0, \infty)$ then M is a closed ball; that is, there are constants $a < b$ with

$$M = \{\phi \in BC | a \leq \phi(t) \leq b, t \geq 0\}.$$

In both cases we ask that for $\phi \in M$ we have

$$(Q\phi)(t) = F(t) + \int_0^t L(t-s)v(s, \phi(s))ds \tag{11}$$

with L completely monotone, $\int_0^\infty L(s)ds = 1$, F uniformly continuous, and $v(t, x)$ bounded and continuous on $[0, T] \times M$. Here, $I = [0, T]$ with T finite or $+\infty$.

Theorem 3.1 *Let M, BC, v, F, and Q be defined in the sentence with (11). If there is a $K > 0$ such that $|v(t, \phi(t))| \leq K$ for $t \in I$ and $\phi \in M$, if Q is continuous on M, and if $Q : M \to M$ then Q has a fixed point in M.*

This theorem is a consequence of a result in the next section. It is isolated here so that its relation to (10) is clear and its simplicity is apparent. Display (8) is the critical property and we now give a quick and simple example illustrating its role in continuity and boundedness.

Example *Let $v(t, x) = V(x)$ and suppose that there is a $K > 0$ so that V is continuous for $|x| \leq K$. Moreover, suppose that M is the the closed $K-$ball of BC and $Q : M \to BC$. We will show that Q is continuous. As V is continuous on $-K \leq x \leq K$, it is uniformly continuous. For each $\epsilon > 0$ there is a $\delta > 0$ so that $x, y \in [-K, K]$ and $|x - y| < \delta$ implies that $|V(x) - V(y)| < \epsilon$. Thus, if $\phi, \psi \in M$ and $\|\psi - \phi\| < \delta$ we have $|V(\phi(t)) - V(\psi(t))| < \epsilon$ for $t \geq 0$. Hence,*

$$|(Q\phi)(t) - (Q\psi)(t)| \leq \int_0^t R(t-s)|V(\phi(s)) - V(\psi(s))|ds \leq \epsilon \int_0^t R(t-s)ds < \epsilon$$

so $\|Q\phi - Q\psi\| < \epsilon$.

4. The detailed theorem

This theorem refers to the notation of the last section. When $I = [0, \infty)$ we will also need a Banach space with a weighted norm which we define as follows. Let $g : [0, \infty) \to \Re$, $g(0) = 1$, $g \in \uparrow +\infty$ as $t \to \infty$, and let $(W, |\cdot|_g)$ be the Banach space of continuous functions $\phi : [0, \infty) \to \Re$ for which

$$|\phi|_g := \sup_{0 \leq t < \infty} \frac{|\phi(t)|}{g(t)} \quad \text{exists.}$$

We note that the result here is not particularly unusual in the case $T < \infty$. Indeed, something like this is used in the most standard existence theorems for ordinary differential equations, although T is usually very small. But the case of $T = \infty$ is most unusual.

Theorem 4.1 *Let the conditions with (11) hold and let $(BC, \|\cdot\|)$ be the Banach space of all continuous functions $\phi : I \to \Re$ for which the supremum norm exists.*

(i) Let $I = [0, T]$ with $T < \infty$, let $M \subset BC$ be bounded and let $v(t, \phi(t))$ be bounded for $\phi \in M$. Then QM is contained in a compact subset of $(BC, \|\cdot\|)$. If, in addition, Q is continuous on M with M closed, convex, and nonempty, then Q has a fixed point in M.

(ii) If $I = [0, \infty)$, if $M \subset BC$ is bounded, and if $|v(t, \phi(t))|$ is bounded for $\phi \in M$, then QM is equicontinuous.

(iii) Let (ii) hold and let $M = \{\phi \in BC | a \leq \phi(t) \leq b, t \geq 0\}$ with $Q : M \to M$. Then QM is contained in a compact subset of $(W, |\cdot|_g)$. If, in addition, Q is continuous on M in the supremum norm, then Q has a fixed point in M.

Proof. We begin with (ii) and point out that it was shown in [4] that under these conditions QM is an equicontinuous set when $T = \infty$. The same is certainly true for $I = [0, T]$ for $T < \infty$; moreover, by Ascoli's theorem, QM resides in a compact set.

Moving back to (i) we then see that Q is a continuous map of M into a compact subset of M and so it has a fixed point by Schauder's theorem.

For (iii), it is shown in detail in [5] that QM resides in a compact subset of $(W, |\cdot|_g)$; a parallel proof can be found in [2, p. 169]. The set M is also closed in the weighted norm. To see that M is closed in the weighted space, suppose that $\{\phi_n\}$ is a sequence residing in M and that sequence converges to ψ so that

$$\sup_{0 \leq t < \infty} \frac{|\phi_n(t) - \psi(t)|}{g(t)} \to 0$$

as $n \to \infty$. Notice that on compact subsets the sequence converges uniformly in the supremum norm to ψ and so ψ is continuous. We must show that ψ resides entirely in M. If it does not, then there is a t_1 with $\psi(t_1)$ outside the closed interval $[a, b]$ and so there is a $D > 0$ with

$$\frac{|\phi_n(t_1) - \psi(t_1)|}{g(t_1)} \geq D,$$

a contradiction to that quantity tending to zero as $n \to \infty$.

It is shown in [5] that Q continuous on M in the supremum norm implies that Q is continuous on M in the weighted norm. Applying Schauder's theorem in the weighted space completes the proof. The boundedness of M in the supremum norm is crucial and makes everything work.

5. Ignore initial conditions

Some simple observations can greatly ease the work of showing that $P : M \to M$. In so many problems [4], [5], and [6] we contrive a set M with $|\phi(t) - u(t, \phi(t))| \leq 1$ for $\phi \in M$. In this case we would like to take $M = \{\phi | \|\phi\| \leq 1\}$. Our mapping is

$$(P\phi)(t) = x(0)[1 - \int_0^t R(s)ds] + \int_0^t R(t-s)[\phi(s) - u(s, \phi(s))]ds.$$

It really seems that the initial condition will ruin the mapping. This is safe for $|x(0)| \leq 1$ because we then have $\phi \in M$ implies that

$$|(P\phi)(t)| \leq [1 - \int_0^t R(s)ds] + \int_0^t R(t-s)ds = 1;$$

it is true that $P : M \to M$.

6. Extend the set of problems

Suppose we have a scalar functional differential equation (ordinary derivative)

$$x'(t) = F(t, x(t), x(t-h)),$$

and a given continuous initial function $\psi : [-h, 0] \to \Re$ with $x(t) = \psi(t)$ for $-h \leq t \leq 0$. See [2] for theory of such equations. Integrate the equation and obtain

$$x(t) = \psi(0) + \int_0^t F(s, x(s), x(s-h))ds.$$

This is the standard step in existence theory but the integral would grow in such an uncontrolled way that fixed point theory would be restricted to a short interval [2, p. 183]. Exactly as in (5), multiply and divide by $J > 0$, subtract and add $x(s)$ to obtain

$$x(t) = \psi(0) + \int_0^t J\left[-x(s) + x(s) + \frac{F(s, x(s), x(s-h))}{J}\right]ds.$$

The kernel is J and we separate out the linear part, form the resolvent, find $R(t) = Je^{-Jt}$ with the property that

$$\int_0^t Je^{-Js}ds = 1 - e^{-Jt} \to 1$$

as $t \to \infty$. Then

$$x(t) = \psi(0)[1 - \int_0^t R(s)ds] + \int_0^t R(t-s)\left[x(s) - \frac{F(s, x(s), x(s-h))}{J}\right]ds,$$

where $x(t) = \psi(t)$ for $-h \leq t \leq 0$. Now R will control the growth at the "expense" of the added $x(s)$ in the integrand. That "expense" turns out to be an asset, as seen throughout [4], [5], and [6]. It is all set up for our fixed point theory exactly as (9). Each problem draws us to a form of (9) which is especially fixed point friendly.

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