A NOTE ON STABILITY BY SCHAUDER'S THEOREM

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Abstract. In this paper we consider a nonlinear perturbation of an asymptotically stable linear differential equation. The standard perturbation theory using Gronwall type inequalities or Liapunov's direct method do not seem to yield asymptotic stability of the perturbed equation. Our purpose here is to give an example of a stability result using fixed point theory.

0. Introduction

Consider a system

$$x' = A(t)x + B(t, x) \tag{a}$$

where solutions of

$$y' = A(t)y \tag{b}$$

are known to be bounded and, perhaps, tend to zero as $t \to \infty$, while $B(t, x)$ is small compared to x in some sense for small x. There is a large classical theory concerning the asymptotic behavior of solutions of (a). An early account is found in Chapters 2 and 3 of Bellman [1], with progressive treatments in Coddington and Levinson [2], Hartman [4], Yoshizawa [7], and Hale [3], to name just a few. Frequently it is assumed that either:

i) The zero solution of (b) is uniformly asymptotically stable, or

ii) A is constant or periodic and all solutions of (b) are bounded.

Then the additional assumption that $|B(t,x)|/|x| \leq g(t)$ where $\int_0^\infty g(t)dt < \infty$ can often be used to conclude that the zero solution of (a) is stable and, possibly, that small solutions tend to zero. If B is independent of t, much more can be said.

To place our work here in perspective, the reader might consider the following result found in Bellman [1;p. 91].

THEOREM. Let A be a constant $n \times n$ matrix, $f(t,x)$ be continuous, all solutions of $y' = Ay$ be bounded, and let

$$|f(t,x)|/|x| \leq c_1 g(t) \text{ where } \int_0^\infty g(t)dt < \infty.$$

Then the zero solution of $x' = Ax + f(t,x)$ is stable.

Methods of proof usually involve Liapunov functions or Gronwall type inequalities which depend so much on properties i) and ii) above.

In a series of projects we are investigating ways in which fixed point theory can be used to obtain stability results which have eluded investigators using the above mentioned methods. Thus, we are particularly interested in examples and fine detail. Here, we consider an equation

$$x'' + 2f(t)x' + x + g(t)x^2 = 0, \quad t \in R^+, \tag{1}$$

with a prototype being $f(t) = g(t) = 1/(t+1)$. The linear part is asymptotically stable, but not uniformly asymptotically stable; and this makes it difficult to obtain asymptotic stability for a perturbed system. Moreover, $g(t)$ is too big for most results since it is not integrable to infinity.

The question we propose to answer here is: "How can we effectively use fixed point theory to prove that the zero solution of (1) is asymptotically stable?"

1. Asymptotic Stability

Consider the scalar equation

$$x'' + 2f(t)x' + x + g(t)x^2 = 0, \quad t \in R^+, \tag{1}$$

where $R^+ := [0, \infty)$, $f(t)$ and $g(t)$ are continuous, $f(t) > 0$, $f(t) \to 0$ as $t \to \infty$, $\int_0^t f(s)ds \to \infty$ as $t \to \infty$, and

$$|f'(t) + f^2(t)| \leq Kf(t), t \in R^+, K < 1, \tag{2}$$

$$|g(t)| \leq Mf(t), t \in R^+, \tag{3}$$

and where K and M are constant. The linear part of equations such as (1) has been extensively studied, as may be seen in Hatvani [5], for example.

Change (1) to a system

$$\begin{aligned} x' &= y - f(t)x \\ y' &= (f'(t) + f^2(t) - 1)x - f(t)y - g(t)x^2 \end{aligned}$$

and write it as

$$X' = \begin{pmatrix} -f(t) & 1 \\ -1 & -f(t) \end{pmatrix} X + \begin{pmatrix} 0 & 0 \\ f'(t) + f^2(t) & 0 \end{pmatrix} X + \begin{pmatrix} 0 \\ -g(t)x^2 \end{pmatrix},$$

or

$$X' = A(t)X + B(t)X + F(t, X). \tag{4}$$

Notice that

$$A(t) \int_0^t A(s)ds = (\int_0^t A(s)ds)A(t).$$

Hence, the principal matrix solution of $z' = A(t)z$ is

$$\exp(\int_0^t A(s)ds) = \exp(-\int_0^t f(s)ds)\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}.$$

If $U(t) = (u_{ij}(t))$, then we define the norm by the maximum of the row sums of $(|u_{ij}(t)|)$ and we denote that norm by $\|U(t)\|$. We then see that the norms of the matrices in (4) satisfy

$$(5) \qquad \|\exp(\int_s^t A(u)du)\| \leq \sqrt{2}\exp(-\int_s^t f(s)ds), t \geq s \geq 0$$

and

$$\|B(t)\| = |f'(t) + f^2(t)| \leq Kf(t), t \in R^+.$$

Now the solution $X(t)$ of (4) with $X(0) = X_0$ is

$$X(t) = e^{\int_0^t A(s)ds}(X_0 + \int_0^t e^{-\int_0^s A(u)du}(B(s)X(s) + F(s, X(s))ds).$$

We will use Schauder's first theorem to prove that for each small X_0, a solution through X_0 tends to zero as $t \to \infty$. A statement of Schauder's theorem can be found in Smart [6;p.15]. For reference it may be stated as follows.

THEOREM. Let $(C, \|\cdot\|)$ be a normed space, and let S be a compact convex nonempty subset of C. Then every continuous mapping of S into S has a fixed point.

Here, we will take C to be the Banach space of bounded and continuous functions $\phi : R^+ \to R^2$ with the supremum norm, $\|\phi\|$ (which will cause no confusion with the matrix norm given above).

Let a be a number with $0 < a < (1-K)/M$ and let $|x_0| \leq a$, and define

$$S_0 := \{\phi : R^+ \to R^2 | \phi(0) = x_0, |\phi(t)| \leq q(t) \text{ on } R^+, \phi \in C\},$$

where $|\cdot|$ denotes the Euclidean norm on R^2 and

$$q(t) := (1-K)a/(Ma + (1-K-Ma)\exp((1-K)\int_0^t f(s)ds)).$$

Define a map P on S_0 by

$$(P\phi)(t) := e^{\int_0^t A(s)ds}(x_0 + \int_0^t e^{-\int_0^s A(u)du}(B(s)\phi(s) + F(s, \phi(s))ds),$$

and maps P_i for $i = 1, 2, 3$ by

$$(P_1\phi)(t) := e^{\int_0^t A(s)ds} x_0,$$

$$(P_2\phi)(t) := e^{\int_0^t A(s)ds} \int_0^t e^{-\int_0^s A(u)du} B(s)\phi(s)ds,$$

and

$$(P_3\phi)(t) := e^{\int_0^t A(s)ds} \int_0^t e^{-\int_0^s A(u)du} F(s, \phi(s))ds.$$

Note that $(P_1\phi)(0) = x_0$ and $(P_2\phi)(0) = (P_3\phi)(0) = 0$.

Next, we have the following two results.

LEMMA 1. If $\phi \in S_0$ then $|(P\phi)(t)| \leq q(t), t \in R^+$.

Proof. Let $\phi(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$. Then, for $t \in R^+$ we have

$$\begin{aligned}
|(P_1\phi)(t)| =& |e^{\int_0^t A(s)ds}\phi(0)| = \left| e^{-\int_0^t f(s)ds} \begin{pmatrix} x(0)\cos t + y(0)\sin t \\ -x(0)\sin t + y(0)\cos t \end{pmatrix} \right| \\
=& e^{-\int_0^t f(s)ds}|\phi(0)| \\
\leq& ae^{-\int_0^t f(s)ds}.
\end{aligned}$$

Next, for $t \in R^+$ we obtain

$$\begin{aligned}
|(P_2\phi)(t)| =& \left| \int_0^t e^{\int_s^t A(u)du} B(s)\phi(s)ds \right| \\
=& \left| \int_0^t e^{\int_s^t A(u)du} \begin{pmatrix} 0 \\ 1 \end{pmatrix} (f'(s) + f^2(s))x(s)ds \right| \\
=& \left| \int_0^t e^{-\int_s^t f(u)du} \begin{pmatrix} \sin(t-s) \\ \cos(t-s) \end{pmatrix} (f'(s) + f^2(s))x(s)ds \right| \\
\leq& K \int_0^t e^{-\int_s^t f(u)du} f(s)|x(s)|ds.
\end{aligned}$$

Similarly, for $t \in R^+$ we have

$$\begin{aligned}|(P_3\phi)(t)| =&\left|\int_0^t e^{\int_s^t A(u)du} F(s,\phi(s))ds\right| \\ =&\left|\int_0^t e^{-\int_s^t f(u)du} \begin{pmatrix}\sin(t-s)\\ \cos(t-s)\end{pmatrix} g(s)x^2(s)ds\right| \\ \leq& M\left|\int_0^t e^{-\int_s^t f(u)du} f(s)x^2(s)ds\right|.\end{aligned}$$

Recall that $x(t)$ is the first component of $\phi(t)$ and in the definition of S_0 we have $|\phi(t)| \leq q(t)$ so $|x(t) \leq |\phi(t)| \leq q(t)$, and this will now be used.

For $t \in R^+$ we obtain

$$\begin{aligned}|(P\phi)(t)| \leq& ae^{-\int_0^t f(s)ds} + K\int_0^t e^{-\int_s^t f(u)du} f(s)|x(s)|ds + M\int_0^t e^{-\int_s^t f(u)du} f(s)x^2(s)ds \\ \leq& ae^{-\int_0^t f(s)ds} + K\int_0^t e^{-\int_s^t f(u)du} f(s)q(s)ds + M\int_0^t e^{-\int_s^t f(u)du} f(s)q^2(s)ds \\ =&: r(t).\end{aligned}$$

It is easily verified that $q(t)$ is the unique solution of the initial value problem

$$x' = f(t)(K-1+Mx)x, x(0) = a,$$

and $q(t)$ satisfies $q'(t) = -f(t)q(t) + f(t)q(t)(K + Mq(t))$. Thus, we have

$$\begin{aligned}r(t) =& e^{-\int_0^t f(s)ds} + \int_0^t e^{-\int_s^t f(u)du}(Kf(s)q(s) + Mf(s)q^2(s))ds \\ =& ae^{-\int_0^t f(s)ds} + \int_0^t e^{-\int_0^t f(u)du}(q'(s) + f(s)q(s))ds\end{aligned}$$

from the equation for q'. If we integrate $\int_0^t e^{-\int_0^t f(u)du} q'(s)ds$ by parts we get $r(t) = q(t)$ on R^+, which gives the desired inequality.

LEMMA 2. There is a continuous increasing function $\delta = \delta(\epsilon) : (0, 2a) \to (0, \infty)$ with

(6) $$|q(t_0) - q(t_1)| \leq \epsilon \text{ if } 0 \leq t_0 < t_1 < t_0 + \delta$$

and

$$(7) \qquad |(P\phi)(t_0) - (P\phi)(t_1)| \le \epsilon \text{ if } \phi \in S_0 \text{ and } 0 \le t_0 < t_1 < t_0 + \delta.$$

Proof. First, it is easy to see that for any ϵ with $0 < \epsilon < 2a$ there is a $\delta_1 > 0$ such that

$$(8) \qquad |q(t_0) - q(t_1)| \le \epsilon \text{ if } 0 \le t_0 < t_1 < t_0 + \delta_1.$$

Next, for any $\phi \in S_0$ we have

$$(P_1\phi)'(t) = A(t)e^{\int_0^t A(s)ds}x_0.$$

Let G be a number with $0 < f(t) \le G$ on R^+. Then from (5) with $s = 0$ we obtain

$$\begin{aligned} |(P_1\phi)'(t)| \le & \|A(t)\| \|e^{\int_0^t A(s)ds}\| |x_0| \\ \le & \sqrt{2}(1+G)e^{-\int_0^t f(s)ds}a \\ \le & \sqrt{2}(1+G)a. \end{aligned}$$

Thus, for any ϵ with $0 < \epsilon < 2a$ there is a $\delta_2 > 0$ such that

$$(9) \qquad |(P_1\phi)(t_0) - (P_1\phi)(t_1)| \le \epsilon \text{ if } 0 \le t_0 < t_1 < t_0 + \delta_2.$$

Now let $T > 1$ be a number such that

$$q(t) \le \epsilon/6 \text{ if } t \ge T - 1.$$

Then, since we have $|(P_i\phi)(t)| \le q(t), (i = 2, 3)$, it is easy to see that

$$(10_i) \qquad |(P_i\phi)(t_0) - (P_i\phi)(t_1)| \le \epsilon/3 \text{ if } T - 1 \le t_0 < t_1.$$

For any t_0 and t_1 with $0 \le t_0 < t_1 \le T$ we have

$$|(P_2\phi)(t_0) - (P_2\phi)(t_1)|$$

$$
\begin{aligned}
=&|\int_0^{t_0} e^{\int_s^{t_0} A(u)du} B(s)\phi(s)ds - \int_0^{t_1} e^{\int_s^{t_1} A(u)du} B(s)\phi(s)ds| \\
\leq&|\int_0^{t_0} (e^{\int_s^{t_0} A(u)du} - e^{\int_s^{t_1} A(u)du}) B(s)\phi(s)ds| + |\int_{t_0}^{t_1} e^{\int_s^{t_1} A(u)du} B(s)\phi(s)ds| \\
\leq& \int_0^{t_0} \|e^{\int_s^{t_0} A(u)du} - e^{\int_s^{t_1} A(u)du}\| \|B(s)\| |\phi(s)|ds + \int_{t_0}^{t_1} \|e^{\int_s^{t_1} A(u)du}\| \|B(s)\| |\phi(s)|ds \\
\leq& Ka \int_0^{t_0} \|e^{\int_s^{t_0} A(u)du} - e^{\int_s^{t_1} A(u)du}\| f(s)ds + \sqrt{2}Ka \int_{t_0}^{t_1} e^{-\int_s^{t_1} f(u)du} f(s)ds \\
\leq& GKa \int_0^{t_0} \|e^{\int_s^{t_0} A(u)du} - e^{\int_s^{t_1} A(u)du}\| ds + \sqrt{2}GKa|t_0 - t_1|.
\end{aligned}
$$

For any η with $0 < \eta < T$, let

$$
d(\eta) := \sup\{\|e^{\int_s^{t_0} A(u)du} - e^{\int_s^{t_1} A(u)du}\| | 0 \leq s \leq t_0 < t_1 \leq T \text{ and } t_1 \leq t_0 + \eta\}.
$$

It is clear that $d(\eta) \to 0+$ as $\eta \to 0+$. Let δ_3 be a number such that $0 < \delta_3 < 1$, and that $d(\delta_3) \leq \epsilon/(6GKTa)$ and $\delta_3 \leq \epsilon/(6\sqrt{2}GKa)$. Then we have

$$
|(P_2\phi)(t_0) - (P_2\phi)(t_1)| \leq \epsilon/3 \text{ if } 0 \leq t_0 < t_1 \leq T \text{ and } t_1 < t_0 + \delta_3,
$$

which, together with (10_2), yields

$$
|(P_2\phi)(t_0) - (P_2\phi)(t_1)| \leq \epsilon/3 \text{ if } 0 \leq t_0 < t_1 < t_0 + \delta_3. \tag{11}
$$

Similarly, for any t_0 and t_1 with $0 \leq t_0 < t_1 \leq T$ we have

$$
|(P_3\phi)(t_0) - (P_3\phi)(t_1)|
$$

$$
\begin{aligned}
=&|\int_0^{t_0} e^{\int_s^{t_0} A(u)du} F(s,\phi(s))ds - \int_0^{t_1} e^{\int_s^{t_1} A(u)du} F(s,\phi(s))ds| \\
\leq& \int_0^{t_0} \|e^{\int_s^{t_0} A(u)du} - e^{\int_s^{t_1} A(u)du}\| |g(s)| x^2(s)ds + \int_{t_0}^{t_1} \|e^{\int_s^{t_1} A(u)du}\| |g(s)| x^2(s)ds \\
\leq& GMa^2 \int_0^{t_0} \|e^{\int_s^{t_0} A(u)du} - e^{\int_s^{t_1} A(u)du}\| ds + \sqrt{2}Ma^2 \int_{t_0}^{t_1} e^{-\int_s^{t_1} f(u)du} f(s)ds \\
\leq& GMa^2 \int_0^{t_0} \|e^{\int_s^{t_0} A(u)du} - e^{\int_s^{t_1} A(u)du}\| ds + \sqrt{2}GMa^2|t_0 - t_1|.
\end{aligned}
$$

Finally, let δ_4 be a number such that $0 < \delta_4 < 1$, and that

$$d(\delta_4) \le \epsilon/(6GMTa^2) \text{ and } \delta_4 \le \epsilon/(6\sqrt{2}GMa^2).$$

Then we obtain

$$|(P_3\phi)(t_0) - (P_3\phi)(t_1)| \le \epsilon/3 \text{ if } 0 \le t_0 < t_1 \le T \text{ and } t_1 \le t_0 + \delta_4,$$

which together with $(10)_3$ yields

$$(12) \qquad |(P_3\phi)(t_0) - (P_3\phi)(t_1)| \le \epsilon/3 \text{ if } 0 \le t_0 < t_1 < t_0 + \delta_4.$$

Thus, from (8), (9), (11) and (12), for $\delta_5 := \min\{\delta_i : 1 \le i \le 4\}$, we have (6) and (7) with $\delta = \delta_5$. Since we may assume that $\delta_5(\epsilon)$ is nondecreasing, we can easily conclude that there is a continuous increasing function $\delta : (0, 2a) \to (0, \infty)$ which satisfies (6) and (7).

Let S be a set of functions $\phi \in S_0$ such that for the function δ in Lemma 2,

$$|\phi(t_0) - \phi(t_1)| \le \epsilon \text{ if } 0 \le t_0 < t_1 < t_0 + \delta.$$

Then we have the following two lemmas.

LEMMA 3. The set S is a compact convex nonempty subset of C.

Proof. Since the function $\phi(t) := (q(t)/a)x_0$ is contained in S, S is nonempty. Clearly S is a convex subset of C. In order to prove the compactness of S, let $\{\phi_k\}$ be uniformly bounded and equicontinuous on R^+. Thus, considering intervals $[0, n]$, n a positive integer, and using a diagonalization process there is a subsequence, say $\{\phi_k\}$ again, converging uniformly on any compact subset of R^+ to some $\phi \in S$. Because $\phi_k(t) \to 0$ as $t \to \infty$, it will now be possible to show that $\|\phi_k - \phi\| \to 0$ as $k \to \infty$. From the definition of $q(t)$, for any $\epsilon > 0$ there is a $T > 0$ with

$$q(t) < \epsilon/2 \text{ if } t \ge T,$$

which yields

$$|\phi_k(t) - \phi(t)| \leq 2q(t) < \epsilon \text{ if } k \in N \text{ and } t \geq T, \tag{13}$$

where N denotes the set of positive integers. On the other hand, since $\{\phi_k(t)\}$ converges to $\phi(t)$ uniformly on $[0, T]$ as $k \to \infty$, for the ϵ there is a $\kappa \in N$ with

$$|\phi_k(t) - \phi(t)| < \epsilon \text{ if } k \geq \kappa \text{ and } 0 \leq t \leq T,$$

which together with (13) implies that $\|\phi_k - \phi\| < \epsilon$ if $k \geq \kappa$. This shows that $\|\phi_k - \phi\| \to 0$ as $k \to \infty$, proving the compactness of S.

LEMMA 4. The map $P : S \to S$ is continuous.

Proof. For any $\phi \in S$, let $\xi := P\phi$. Clearly we have $\xi(0) = x_0$ and $\xi \in C$. Next, from Lemma 1 we obtain

$$|\xi(t)| \leq q(t), \ t \in R^+.$$

Thus, we have $\xi \in S_0$, which together with Lemma 2 implies that $\xi \in S$. Hence, P maps S into S. We need to prove that P is continuous. For any $\phi_i \in S$ $(i = 1, 2)$ and $t \in R^+$ we have

$$|(P\phi_1)(t) - (P\phi_2)(t)|$$

$$\begin{aligned}
&\leq \left| \int_0^t e^{\int_s^t A(u)du} (B(s)(\phi_1(s) - \phi_2(s)) + (F(s, \phi_1(s)) - F(s, \phi_2(s)))ds \right| \\
&\leq \sqrt{2} \int_0^t e^{-\int_s^t f(u)du} |Kf(s) + 2ag(s)||\phi_1(s) - \phi_2(s)|ds \\
&\leq \sqrt{2}(K + 2Ma)\|\phi_1 - \phi_2\| \int_0^t e^{-\int_s^t f(u)du} f(s)ds \\
&\leq \sqrt{2}(K + 2Ma)\|\phi_1 - \phi_2\|,
\end{aligned}$$

which implies that P is continuous.

In view of Schauder's first theorem we have the following result.

THEOREM. Under assumptions (2) and (3), let a be a number with $0 < a < (1-K)/M$, and let $|x_0| \le a$. Then the solution $x(t, x_0)$ of (4) satisfies

$$|x(t,x_0)| \le (1-K)a/[Ma + (1-K-Ma)\exp((1-K)\int_0^t f(s)ds)].$$

We conclude with an example.

EXAMPLE. Consider the equation

$$x'' + [2/(t+1)]x' + x + x^2/(t+1) = 0, t \in R^+. \tag{14}$$

(Thus, $f(t) = g(t) = 1/(t+1)$.) Clearly, $f(t) \to 0$ as $t \to \infty$, $\int_0^\infty f(s)ds = \infty$. Moreover, it is easy to see that (2) holds with $K \in (0,1)$, and that (3) holds with $M = 1$. Change (14) to a system $X' = F(t, X)$ as before. Let a be a number with $0 < a < 1-K$, and let $|X_0| \le a$. By our result, the solution $X(t, X_0)$ satisfies

$$|X(t,X_0)| \le (1-K)a/[a + (1-K-a)(t+1)^{1-K}].$$

References

1. Bellman, R., Stability Theory of Differential Equations, McGraw-Hill, New York, 1953

2. Coddington, E. A. and N. Levinson, Theory of Ordinary Differential Equations, McGraw-Hill, New York, 1955.

3. Hale, Jack K., Ordinary Differential Equations, Wiley, New York, 1969.

4. Hartman, Philip, Ordinary Differential Equations, Wiley, New York, 1964.

5. Hatvani, L., Integral conditions on the asymptotic stability for the damped linear oscillator with small damping, Proc. Amer. Math. Soc. 124(1996), 415-422.

6. Smart, D. R., Fixed Point Theorems, Cambridge Univ. Press, Cambridge, 1974.

7. Yoshizawa, T., Stability Theory by Liapunov's Second Method, Math. Soc. Japan, Tokyo, 1966.