

Bounded and Periodic Solutions of Integral Equations

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Abstract

In this paper we introduce a new method for obtaining boundedness of solutions of integral equations. From the integral equation we form an integrodifferential equation by computing $x' + kx$ to which we apply a Liapunov functional. This can be far more effective than the usual technique of differentiating the equation. The qualitative properties derived from that equation are then transferred to a majorizing function for the integral equation. Schaefer's fixed point theorem is used to conclude that there is a periodic solution. Three kinds of integral equations are studied and they are shown to be related through two examples.

1 Introduction

In this paper we consider several nonlinear scalar integral equations of the form

$$x(t) = a(t) - \int_{\alpha(t)}^t C(t, s)g(x(s))ds \quad (1.1)$$

where $\alpha(t)$ may be zero, $-\infty$, or $t-h$ for some constant $h > 0$. In each of the problems the kernel need not be convex, but the assumption is that there is a constant $k > 0$

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with

$$D(t, s) := C_t(t, s) + kC(t, s) \quad (1.2)$$

convex. For existence theory see [3], [5], [15].

In 1928 Volterra [17] noted that many physical problems were being modeled by integral and integrodifferential equations with convex kernels. Such kernels are natural representations of fading memory. Today we see such models in problems in biology, neural networks, viscoelasticity, nuclear reactors, and many other places. See [4]–[8],[11], [14], [17], [19]–[20] for work on integral equations with convex kernels.

In addition to the natural fading memory, by 1963 there arose another good reason to try to formulate problems with such kernels. In that same paper Volterra had suggested that there might be constructed a Liapunov functional which would yield very precise qualitative properties of solutions, and that it would admit arbitrarily large kernels. This is in marked contrast to so much of the theory which leaves the investigator strapped with Draconian conditions such as $\int_0^t |C(t, s)| ds \leq \gamma < 1$. In 1963 Levin [10] followed Volterra's suggestion and constructed a Liapunov functional for the convolution case (see also [13]) and in 1968 [12] he constructed (1.5) below for

$$x'(t) = - \int_0^t C(t, s)g(x(s))ds, \quad xg(x) > 0 \text{ if } x \neq 0 \quad (1.3)$$

where $C(t, s)$ is convex:

$$C(t, s) \geq 0, C_s(t, s) \geq 0, C_{st}(t, s) \leq 0, C_t(t, s) \leq 0. \quad (1.4)$$

His functional has the form

$$V(t) = 2 \int_0^x g(s)ds + \int_0^t C_s(t, s) \left(\int_s^t g(x(u))du \right)^2 ds + C(t, 0) \left(\int_0^t g(x(u))du \right)^2 \quad (1.5)$$

with derivative along solutions of (1.3) satisfying

$$V'(t) = \int_0^t C_{st}(t, s) \left(\int_s^t g(x(u))du \right)^2 ds + C_t(t, 0) \left(\int_0^t g(x(u))du \right)^2 \leq 0 \quad (1.6)$$

Much work followed in that same vein. See Zhang [19]–[20] for systems.

In 1992 (see [1] and [2]) we constructed parallel Liapunov functions for (1.1), taking into account the various forms of $\alpha(t)$. Here, also, one finds much work following in the same vein. Long before the 1992 work appeared, investigators had differentiated (1.1) to obtain an integrodifferential equation to which they could apply Liapunov's direct method. Miller [15] formally starts his Chapter 6 with such a presentation. Fruitful as that approach has been, it also has significant difficulties. A more refined

approach was introduced in Chapter 9 of [5] and that has been used with success in a number of subsequent papers, each of which features a new advantage to the technique.

In each case, the idea is to form $x' + kx$ from (1.1). For $\alpha = 0$ we have

$$x' + kx = a'(t) + ka(t) - kC(t, t)g(x(t)) - \int_0^t D(t, s)g(x(s))ds \quad (1.7)$$

with D defined in (1.2). There are six important observations.

- (i) $x' + kx$ is a uniformly asymptotically stable operator for $k > 0$.
- (ii) If $C(t, t) \geq 0$ and if $xg(x) > 0$ for $x \neq 0$, then $x' + kx + C(t, t)g(x)$ is an operator of the same, but stronger, type.
- (iii) If C and C_t differ in sign then $D(t, s)$ is smaller than the larger of the two terms. (See [5] and [9].)
- (iv) Under general conditions if C is convex and k is large then D is convex, while the kernel for x' alone has lost its convexity. (See [6].)
- (v) If $C(t, s)$ is not convex while $D(t, s)$ is, then the combined equation (1.7) is the right form to apply Liapunov functionals. (See Example 3.1.)
- (vi) The utility of a Liapunov functional often depends on the separation of its derivative into a difference, say $|p(t)| - |h(x)|$. Using C alone, that can require strong conditions on g , but when using D there is a natural separation [6].

To be fair, one should ask if something has been lost. It has, and it introduces a new problem. The Liapunov functional which we constructed in 1992 for (1.1) works with a more general $g(t, x)$, but the Levin Liapunov functional which we will use on the $x' + kx$ equation needs $g(x)$. It would be so interesting to extend Levin's Liapunov functional to $g(t, x)$ for (1.3).

That is the background and we now move along with some new problems. The first of which is to use a combination of the two Liapunov functionals and the systems (1.1) and (1.7). First we prove the existence of a periodic solution when $\alpha = \infty$. We then study the case of $\alpha = 0$ proving boundedness properties. Finally, we take $\alpha = t - h$ and prove both boundedness and periodicity.

Let $R^+ = [0, \infty)$, $R = (-\infty, \infty)$, and $C(X, Y)$ denote the space of continuous functions $\phi : X \rightarrow Y$. We also denote by $(P_T, \|\cdot\|)$ the Banach space of continuous T -periodic functions $\phi : R \rightarrow R$ with the supremum norm.

For the existence of periodic solutions, we apply Schaefer's fixed point theorem (see below) with $F(x)$ being the right-hand side of (1.1) so that if F has a fixed point, then this fixed point is a periodic solution of (1.1).

Theorem 1.1 (Schaefer [16]). *Let $(P, \|\cdot\|)$ be a normed space, F a continuous mapping of P into P which is compact on each bounded subset of P . Then either*

- (i) the equation $\phi = \lambda F\phi$ has a solution for $\lambda = 1$, or
- (ii) the set of all such solutions ϕ , for $0 < \lambda < 1$, is unbounded.

2 Boundedness and Periodicity

We consider the equation

$$x(t) = \lambda[a(t) - \int_{-\infty}^t C(t, s)g(x(s))ds], \quad 0 \leq \lambda \leq 1 \quad (2.1)$$

where $a : R \rightarrow R$, $C : R \times R \rightarrow R$, $g : R \rightarrow R$ are all continuous. Suppose that there is a positive constant k so that

$$D(t, s) := C_t(t, s) + kC(t, s) \text{ is convex.} \quad (2.2)$$

We first want to show that there exists a constant $\gamma > 0$ such that $|x(t)| \leq \gamma$ whenever x is a T -periodic solution of (2.1) for all $0 \leq \lambda \leq 1$. We then show the existence of a T -periodic solution of (2.1) for $\lambda = 1$ by applying Schaefer's fixed point theorem. Our main assumptions are that there is a $T > 0$ and $J > 0$ such that

$$a(t + T) = a(t), \quad C(t + T, s + T) = C(t, s) \quad (2.3)$$

for all $s \leq t$ with a' continuous and

$$\sup_{0 \leq t \leq T} \int_{-\infty}^t D(t, s)(t - s)ds \leq J \quad (2.4)$$

and that D satisfies

$$D(t, s) \geq 0, \quad D_s(t, s) \geq 0, \quad D_{st}(t, s) \leq 0. \quad (2.5)$$

We differentiate (2.1) and form

$$x' + kx = \lambda[a' + ka - C(t, t)g(x) - \int_{-\infty}^t D(t, s)g(x(s))ds]. \quad (2.6)$$

Now, we define the Liapunov functional

$$V(t) = 2 \int_0^{x(t)} g(s)ds + \lambda \int_{-\infty}^t D_s(t, s) \left(\int_s^t g(x(v))dv \right)^2 ds \quad (2.7)$$

for $x \in (P_T, \|\cdot\|)$.

Theorem 2.1. *Suppose that (2.3), (2.4), and (2.5) hold. If $x(t)$ is a T -periodic solution of (2.1), then the derivative of V along that solution satisfies*

$$V'(t) \leq 2\lambda g(x)[a'(t) + ka(t)] - 2g(x)[kx + \lambda C(t, t)g(x)]. \quad (2.8)$$

If there is an $L > 0$ with

$$xg(x) \geq 0 \text{ for } |x| \geq L \quad (2.9)$$

and if, in addition, there is a $\mu > 0$ with

$$g(x)[kx + C(t, t)g(x)] \geq \mu g^2(x) \quad (2.10)$$

for $|x| \geq L$, then there is an $M > 0$ with

$$V'(t) \leq -|g(x)| + M. \quad (2.11)$$

Proof. We first define some constants to simplify notation. Integrating by parts, we obtain

$$\begin{aligned} \int_b^t D_s(t, s)(t-s)^2 ds &= D(t, s)(t-s)^2 \Big|_b^t + 2 \int_b^t D(t, s)(t-s) ds \\ &= -D(t, b)(t-b)^2 + 2 \int_b^t D(t, s)(t-s) ds \end{aligned}$$

for each $b < t$. Since $D(t, s) \geq 0$ and $D_s(t, s) \geq 0$, letting $b \rightarrow -\infty$, we see that

$$\int_{-\infty}^t D_s(t, s)(t-s)^2 ds + \lim_{s \rightarrow -\infty} D(t, s)(t-s)^2 = 2 \int_{-\infty}^t D(t, s)(t-s) ds \leq 2J. \quad (2.12)$$

Observe also that

$$D(t, b)(t-b)^2 \leq 2 \int_b^t D(t, s)(t-s) ds \leq 2J$$

for all $b \leq t$. This then implies that $D(t, s)(t-s) \leq 2J/(t-s)$ for all $s < t$, and so we arrive at

$$\lim_{s \rightarrow -\infty} (t-s)D(t, s) = 0 \text{ for fixed } t \quad (2.13)$$

and obtain

$$\int_{-\infty}^t D_s(t, s) ds = \lim_{b \rightarrow -\infty} [D(t, t) - D(t, b)] \leq \sup_{0 \leq t \leq T} D(t, t) =: B. \quad (2.14)$$

Now let x be a T -periodic solution of (2.1) and $V(t)$ be defined in (2.7). It follows from (2.12) that $V(t)$ is well-defined and T -periodic. We then find

$$\begin{aligned} V'(t) &= 2g(x)x'(t) + \lambda \int_{-\infty}^t D_{st}(t, s) \left(\int_s^t g(x(v))dv \right)^2 ds \\ &\quad + 2\lambda g(x) \int_{-\infty}^t D_s(t, s) \int_s^t g(x(v))dv ds. \end{aligned}$$

Integration of the last term by parts and use of (2.13) in the lower limit for that periodic solution yields

$$\int_{-\infty}^t D_s(t, s) \int_s^t g(x(v))dv ds = \int_{-\infty}^t D(t, s)g(x(s))ds.$$

Since $D_{st}(t, s) \leq 0$, the second term of V' is not positive, and thus, if we use (2.6), we obtain

$$\begin{aligned} V'(t) &\leq 2g(x) \left[-kx + \lambda(a' + ka) - \lambda C(t, t)g(x) - \lambda \int_{-\infty}^t D(t, s)g(x(s))ds \right] \\ &\quad + 2\lambda g(x) \int_{-\infty}^t D(t, s)g(x(s))ds \\ &= 2\lambda g(x)[a'(t) + ka(t)] - 2g(x)[kx + \lambda C(t, t)g(x)] \end{aligned}$$

verifying (2.8).

Next we choose $N > 1$ so large that $-\mu(N - 1) < C_* = \min\{C(t, t) : 0 \leq t \leq T\}$, where $\mu > 0$ is defined in (2.10). If $|x| \geq L$, then $xg(x) \geq 0$, and by (2.10), we obtain for $|x(t)| \geq L$ that

$$\begin{aligned} V'(t) &\leq 2\lambda|g(x)|[\|a'\| + k\|a\|] - 2g(x)[kx + \lambda C(t, t)g(x)] \\ &\leq 2|g(x)|[\|a'\| + k\|a\|] - 2k\frac{1}{N}xg(x) \\ &\quad - 2\lambda \left(1 - \frac{1}{N}\right) g(x)[kx + C(t, t)g(x)] - 2\lambda\frac{1}{N}C(t, t)g^2(x) \\ &\leq 2|g(x)|[\|a'\| + k\|a\|] - 2k\frac{1}{N}xg(x) \\ &\quad - 2\lambda \left(1 - \frac{1}{N}\right) \mu g^2(x) - 2\lambda\frac{1}{N}C(t, t)g^2(x) \\ &\leq -2|g(x)| \left[\frac{1}{N}k|x| - (\|a'\| + k\|a\|) \right]. \end{aligned}$$

We may assume that $L \geq N(\|a'\| + k\|a\| + 1)/k$. Thus, if $|x(t)| \geq L$, then $V'(t) \leq -|g(x(t))|$. It is clear that $V'(t) \leq M$ for $0 \leq |x(t)| \leq L$, where

$$M = 2g_L[\|a'\| + k\|a\|] + 2g_L[kL + C^*g_L]$$

with $g_L = \sup\{|g(x)| : |x| \leq L\}$ and $C^* = \sup\{|C(t, t)| : 0 \leq t \leq T\}$, and hence,

$$V'(t) \leq -|g(x(t))| + M$$

for all $t \geq 0$. This completes the proof.

To establish an *a priori* bound for all possible T -periodic solutions of (2.1), we assume that

$$\lim_{s \rightarrow -\infty} (t-s)C(t, s) = 0 \quad \text{and} \quad \int_{-\infty}^t |C_s(t, s)|(t-s)ds \leq J_1 \quad (2.15)$$

for $J_1 > 0$.

Theorem 2.2. *Suppose that (2.3)-(2.5), and (2.15) hold. Then there exists a constant $\gamma > 0$ such that $\|x\| < \gamma$ whenever x is a T -periodic solution of (2.1).*

Proof. Let x be a T -periodic solution of (2.1) and $V(t)$ be defined in (2.7). Then (2.11) holds. Since $V(t)$ is T -periodic, $V(t)$ has a global maximum at $q \in [0, T]$ and, hence, at $t_n = q + nT$. So for $s \leq t_n$, we have

$$0 \leq V(t_n) - V(s) \leq -\int_s^{t_n} |g(x(v))|dv + M(t_n - s).$$

and so

$$\int_s^{t_n} |g(x(v))|dv \leq M(t_n - s).$$

Then $(x(t) - \lambda a(t))^2$ has a global maximum at $h_n := t_n + p$, where $0 \leq p \leq T$, and for $s \leq h_n$ we have

$$\int_s^{h_n} |g(x(v))|dv \leq \int_s^{t_{n+1}} |g(x(v))|dv \leq M(t_{n+1} - s).$$

It follows from (2.1) that

$$\begin{aligned} (x(h_n) - \lambda a(h_n))^2 &\leq \left(\int_{-\infty}^{h_n} C(h_n, s)g(x(s))ds \right)^2 \\ &= \left(-C(h_n, s) \int_s^{h_n} g(x(v))dv ds \Big|_{-\infty}^{h_n} + \int_{-\infty}^{h_n} C_s(h_n, s) \int_s^{h_n} g(x(v))dv ds \right)^2 \end{aligned}$$

$$\begin{aligned}
&= \left(\int_{-\infty}^{h_n} C_s(h_n, s) \int_s^{h_n} g(x(v)) dv ds \right)^2 \\
&\leq \left(\int_{-\infty}^{h_n} |C_s(h_n, s)| \int_s^{t_{n+1}} |g(x(v))| dv ds \right)^2 \\
&\leq M^2 \left(\int_{-\infty}^{h_n} |C_s(h_n, s)| (h_n + T - s) ds \right)^2.
\end{aligned}$$

Since $\int_{-\infty}^t |C_s(t, s)| ds$ is T -periodic, we see from (2.15) that

$$\sup_{0 \leq t \leq T} \int_{-\infty}^t |C_s(t, s)| ds \leq J_0 \quad (2.16)$$

for $J_0 > 0$, and hence,

$$\left(x(h_n) - \lambda a(h_n) \right)^2 \leq M^2 (TJ_0 + J_1)^2.$$

Noticing that M is a function of L , we find that

$$|x(h_n)| < \|a\| + M(TJ_0 + J_1) + 1 := \gamma.$$

This implies that $\|x\| < \gamma$ whenever x is a T -periodic solution of (2.1) for $0 \leq \lambda \leq 1$, and the proof is complete.

We now define a mapping F on P_T by

$$F(\phi)(t) = a(t) - \int_{-\infty}^t C(t, s)g(\phi(s))ds \text{ for } \phi \in P_T. \quad (2.17)$$

Theorem 2.3. *If (2.3)-(2.5), and (2.15) hold, then (2.1) has a T -periodic solution for $\lambda = 1$.*

Proof. It is clear that $F(\phi) \in P_T$. We show that F is continuous on P_T and is compact on each bounded subset of P_T . If $\tilde{\phi}, \phi \in P_T$, then

$$\begin{aligned}
|F(\phi)(t) - F(\tilde{\phi})(t)| &= \left| \int_{-\infty}^t C(t, s)g(\phi(s))ds - \int_{-\infty}^t C(t, s)g(\tilde{\phi}(s))ds \right| \quad (2.18) \\
&= \left| \int_{-\infty}^t C_s(t, s) \left(\int_s^t g(\phi(v))dv - \int_s^t g(\tilde{\phi}(v))dv \right) ds \right|.
\end{aligned}$$

Since g is uniformly continuous on $\{x \in R : |x| \leq \|\tilde{\phi}\| + 1\}$, for any $\varepsilon > 0$,

there exists $0 < \delta < 1$ such that $\|\phi - \tilde{\phi}\| < \delta$ implies $|g(\phi(s)) - g(\tilde{\phi}(s))| < \varepsilon$ for all $s \in [0, T]$. It follows from (2.18) that $\|F(\phi) - F(\tilde{\phi})\| \leq J_1\varepsilon$. Thus, F is continuous on P_T .

We now show that F is compact on each bounded subset of P_T . Let $\eta > 0$ and define

$$\Gamma = \{F(\phi) : \phi \in P_T, \|\phi\| \leq \eta\}. \quad (2.19)$$

Since

$$\begin{aligned} \frac{d}{dt}F(\phi)(t) &= a'(t) - C(t, t)g(\phi(t)) - \int_{-\infty}^t C_t(t, s)g(\phi(s))ds \\ &= a'(t) - C(t, t)g(\phi(t)) - \int_{-\infty}^t D(t, s)g(\phi(s))ds + k \int_{-\infty}^t C(t, s)g(\phi(s))ds \\ &= a'(t) - C(t, t)g(\phi(t)) - \int_{-\infty}^t D(t, s)g(\phi(s))ds + k \int_{-\infty}^t C_s(t, s) \int_s^t g(\phi(v))dv ds \end{aligned}$$

we have

$$\left| \frac{d}{dt}F(\phi)(t) \right| \leq \|a'\| + g_\eta \sup_{0 \leq t \leq T} \left(|C(t, t)| + \int_{-\infty}^t D(t, s)ds + k \int_{-\infty}^t |C_s(t, s)|(t-s)ds \right)$$

where $g_\eta = \{|g(x)| : |x| \leq \eta\}$, and thus, Γ is equi-continuous. The uniform boundedness of Γ follows from the inequality

$$|F(\phi)(t)| \leq \|a\| + \int_{-\infty}^t |C_s(t, s)| \int_s^t |g(\phi(v))|dv \leq \|a\| + J_1 g_\eta$$

for all $\phi \in \Gamma$. So, by the Ascoli-Arzelà theorem, Γ lies in a compact subset of P_T . By combining Schaefer's theorem with Theorem 2.2, we see that F has a fixed point which is a T -periodic solution of (2.1) for $\lambda = 1$. This completes the proof.

Corollary 2.1. *Suppose that (2.3)-(2.5) hold. If there is an $L > 0$ and $\mu > 0$ with*

$$xg(x) \geq 0 \text{ for } |x| \geq L \text{ and } C(t, t) \geq \mu, \quad (2.20)$$

then there is an $M > 0$ with

$$V'(t) \leq -|g(x)| + M \quad (2.21)$$

whenever x is a T -periodic solution of (2.1). If, in addition, (2.15) is satisfied, then (2.1) has a T -periodic solution for $\lambda = 1$.

3 Boundedness

We turn now to

$$x(t) = a(t) - \int_0^t C(t, s)g(x(s))ds \quad (3.1)$$

where $a : R^+ \rightarrow R$, $C : R^+ \times R^+ \rightarrow R$, $g : R \rightarrow R$ are all continuous with a, a' bounded. The project here is to show that solutions of (3.1) are bounded. We define

$$D(t, s) := C_t(t, s) + kC(t, s) \quad (3.2)$$

for a constant $k > 0$. Our main assumptions are that

$$D(t, s) \text{ is convex for } t \geq s \geq 0 \quad (3.3)$$

and there exists $B > 0$ with

$$C(t, t) \geq -B \text{ and } \int_0^t D(t, s)(t - s)ds \leq B \quad (3.4)$$

for all $t \geq 0$. We differentiate (3.1) and form

$$x' + kx = [a' + ka - C(t, t)g(x) - \int_0^t D(t, s)g(x(s))ds]. \quad (3.5)$$

Now, we define the Liapunov functional

$$V(t) = 2G(x(t)) + \int_0^t D_s(t, s) \left(\int_s^t g(x(u))du \right)^2 ds + D(t, 0) \left(\int_0^t g(x(u))du \right)^2 \quad (3.6)$$

for $x \in C(R^+, R)$, where $G(x) = \int_0^x g(s)ds$.

Theorem 3.1. *Suppose that $D(t, s)$ is convex and $C(t, t) \geq -B$ for a constant $B > 0$. If $x(t)$ is a solution of (3.1), then the derivative of V along that solution satisfies*

$$V'(t) \leq 2g(x)[a'(t) + ka(t)] - 2g(x)[kx + C(t, t)g(x)]. \quad (3.7)$$

If there is an $L > 0$ with

$$xg(x) \geq 0 \text{ for } |x| \geq L \quad (3.8)$$

and if, in addition, there is a $\mu > 0$ with

$$g(x)[kx + C(t, t)g(x)] \geq \mu g^2(x) \quad (3.9)$$

for $|x| \geq L$, then there is an $M > 0$ with

$$V'(t) \leq -|g(x)| + M. \quad (3.10)$$

Proof. We first observe that if x is a solution of (3.1), then x is also a solution of (3.5). Now let x be a solution (3.1) and $V(t)$ be defined in (3.6). We then find

$$\begin{aligned} V'(t) &= 2g(x)x'(t) + \int_0^t D_{st}(t, s) \left(\int_s^t g(x(u)) du \right)^2 ds \\ &\quad + 2g(x(t)) \int_0^t D_s(t, s) \int_s^t g(x(u)) dud s + D_t(t, 0) \left(\int_0^t g(x(u)) du \right)^2 \\ &\quad + 2D(t, 0)g(x(t)) \int_0^t g(x(u)) du. \end{aligned}$$

Integrate the third to last term by parts to obtain

$$\begin{aligned} &2g(x(t)) \int_0^t D_s(t, s) \int_s^t g(x(u)) dud s \\ &= 2g(x(t)) \left[D(t, s) \int_s^t g(x(u)) du \Big|_{s=0}^{s=t} + \int_0^t D(t, s)g(x(s)) ds \right] \\ &= 2g(x(t)) \left[-D(t, 0) \int_0^t g(x(s)) ds + \int_0^t D(t, s)g(x(s)) ds \right]. \quad (3.11) \end{aligned}$$

Cancel terms, use the sign conditions, and use (3.5) in the process to unite the Liapunov functional and the equation to obtain

$$\begin{aligned} V'(t) &\leq 2g(x) \left[-kx + (a' + ka) - C(t, t)g(x) - \int_0^t D(t, s)g(x(s)) ds \right] \\ &\quad + 2g(x) \int_0^t D(t, s)g(x(s)) ds \\ &= 2g(x)[a'(t) + ka(t)] - 2g(x)[kx + C(t, t)g(x)] \end{aligned}$$

verifying (3.7).

Now we assume that (3.8) and (3.9) hold. We may choose $N > 1$ so large that $\mu(N - 1) > B$, where B and μ are defined in (3.4) and (3.9), respectively. If $|x| \geq L$, then $xg(x) \geq 0$, and by (3.9), we obtain for $|x(t)| \geq L$ that

$$\begin{aligned} V'(t) &\leq 2|g(x)|[|a'| + k|a|] - 2g(x)[kx + C(t, t)g(x)] \\ &= 2|g(x)|[|a'| + k|a|] - 2k\frac{1}{N}xg(x) \\ &\quad - 2\left(1 - \frac{1}{N}\right)g(x)[kx + C(t, t)g(x)] - 2\frac{1}{N}C(t, t)g^2(x) \end{aligned}$$

$$\begin{aligned}
&\leq 2|g(x)|[|a'| + k|a|] - 2k\frac{1}{N}xg(x) \\
&\quad - 2\left(1 - \frac{1}{N}\right)\mu g^2(x) + 2\frac{1}{N}Bg^2(x) \\
&\leq -2|g(x)|\left[\frac{1}{N}k|x| - (|a'(t)| + k|a(t)|)\right].
\end{aligned}$$

We may assume that

$$L \geq N[\sup_{t \geq 0}(|a'(t)| + k|a(t)|) + 1]/k.$$

Thus, if $|x(t)| \geq L$, then $V'(t) \leq -|g(x(t))|$. Since $-C(t, t) \leq B$, it is clear that $V'(t) \leq M$ for $0 \leq |x(t)| \leq L$, where the constant $M > 0$ is a function of L , and hence,

$$V'(t) \leq -|g(x(t))| + M$$

for all $t \geq 0$. This completes the proof.

Relations related to (3.9) and (3.10) are found in [4].

To establish the boundedness of solutions, we assume that there is a $B_1 > 0$ with

$$|C(t, 0)|t \leq B_1 \quad \text{and} \quad \int_0^t |C_s(t, s)|(t - s + 1)ds \leq B_1 \quad (3.12)$$

for $t \geq 0$. We also observe that

$$\begin{aligned}
\int_0^t D_s(t, s)(t - s)^2 ds &= D(t, s)(t - s)^2 \Big|_0^t + 2 \int_0^t D(t, s)(t - s) ds \\
&= -D(t, 0)t^2 + 2 \int_0^t D(t, s)(t - s) ds.
\end{aligned}$$

By (3.4), we now have

$$\int_0^t D_s(t, s)(t - s)^2 ds + D(t, 0)t^2 = 2 \int_0^t D(t, s)(t - s) ds \leq 2B \quad (3.13)$$

for all $t \geq 0$.

Theorem 3.2. *If (3.3)-(3.4), (3.8)-(3.9), and (3.12) hold, then any solution of (3.1) is bounded.*

Proof. Let x be a solution of (3.1) and $V(t)$ be defined in (3.6). Then $V(t)$ is bounded below and satisfies (3.10). We now show that $V(t)$ is bounded above. If

$V(t)$ is unbounded, then there exists a sequence $\{t_n\} \uparrow \infty$ with $V(t_n) \rightarrow \infty$ as $n \rightarrow \infty$ and

$$V(t_n) \geq V(s) \quad \text{for } 0 \leq s \leq t_n.$$

It then follows from (3.10) that

$$0 \leq V(t_n) - V(s) \leq - \int_s^{t_n} |g(x(u))| du + M(t_n - s).$$

This implies that

$$\int_s^{t_n} |g(x(u))| du \leq M(t_n - s). \quad (3.14)$$

Applying (3.14) to $V(t_n)$ and taking into account (3.13), we find that

$$\begin{aligned} V(t_n) &\leq 2G(x(t_n)) + M^2 \left[\int_0^{t_n} D_s(t_n, s)(t_n - s)^2 ds + D(t_n, 0) t^2 \right] \\ &\leq 2G(x(t_n)) + 2BM^2. \end{aligned} \quad (3.15)$$

We now use (3.1), (3.12), and (3.14) to obtain

$$\begin{aligned} (x(t_n) - a(t_n))^2 &= \left(\int_0^{t_n} C(t_n, s)g(x(s))ds \right)^2 \\ &= \left(-C(t_n, s) \int_s^{t_n} g(x(u))du \Big|_{s=0}^{s=t_n} + \int_0^{t_n} C_s(t_n, s) \int_s^{t_n} g(x(u))duds \right)^2 \\ &= \left(C(t_n, 0) \int_0^{t_n} g(x(s))ds + \int_0^{t_n} C_s(t_n, s) \int_s^{t_n} g(x(u))duds \right)^2 \\ &\leq \left(|C(t_n, 0)|M t_n + M \int_0^{t_n} |C_s(t_n, s)|(t_n - s)ds \right)^2 \leq M^2(2B_1)^2. \end{aligned}$$

This implies that

$$|x(t_n)| \leq \sup_{s \geq 0} |a(s)| + 2B_1M := B_2 \quad (3.16)$$

and that $|G(x(t_n))| \leq B_3$ for a $B_3 > 0$. We now find that

$$V(t_n) \leq 2G(x(t_n)) + 2BM^2 \leq 2B_3 + 2BM^2 := B_4,$$

a contradiction. Thus, $V(t)$ is bounded. In fact, we have

$$2G(x(t)) \leq V(t) \leq 2G(x(0)) + B_4 \quad (3.17)$$

and hence

$$|V(t)| \leq K \text{ for all } t \geq 0 \quad (3.18)$$

where $K := 2|\eta| + 2|G(x(0))| + B_4$ where $\eta = \inf\{G(u) : u \in R\}$. We also observe that $|C(t, 0)| \leq B_5$ for a $B_5 > 0$ whenever (3.12) holds.

We now integrate (3.10) from s to t and use (3.18) to obtain

$$\int_s^t |g(x(u))| du \leq V(s) - V(t) + M(t - s) \leq 2K + M(t - s). \quad (3.19)$$

Applying (3.19) to (3.1) we find

$$\begin{aligned} |x(t)| &\leq |a(t)| + \left| \int_0^t C(t, s)g(x(s))ds \right| \\ &\leq |a(t)| + \left| C(t, 0) \int_0^t g(x(s))ds + \int_0^t C_s(t, s) \int_s^t g(x(u))duds \right| \\ &\leq |a(t)| + |C(t, 0)|(Mt + 2K) + \int_0^t |C_s(t, s)|[M(t - s) + 2K]ds \\ &\leq \sup_{s \geq 0} |a(s)| + B_1M + 2KB_5 + B_1(M + 2K). \end{aligned}$$

This implies that x is bounded. The proof is complete.

Corollary 3.1. *Suppose that (3.3)-(3.4) hold. If there is an $L > 0$ and $\mu > 0$ with*

$$xg(x) \geq 0 \text{ for } |x| \geq L \text{ and } C(t, t) \geq \mu, \quad (3.20)$$

then there is an $M > 0$ with

$$V'(t) \leq -|g(x)| + M \quad (3.21)$$

whenever x is a solution of (3.1). If, in addition, (3.12) holds, then $V(t)$ satisfies (3.17) and any solution of (3.1) is bounded.

Remark 3.1. Inequalities related to (3.17) and (3.21) are of fundamental importance in the study of boundedness and periodic solutions in differential equations by Liapunov's direct method (see Burton [3] and Yoshizawa [18]). Not only are these practical inequalities with many applications, but such combined relations are directly linked to the right-hand side of the equations, and hence, much of the qualitative properties of solutions can be derived by taking full advantage of the Liapunov functions.

The following example shows that if $C(t, s)$ is not convex while $D(t, s)$ is, then the combined equation (1.7) is the right form to apply Liapunov functionals.

Example 3.1. We consider the equation

$$x(t) = a(t) - \int_0^t C(t, s)g(x(s))ds \quad (3.22)$$

where $a : R^+ \rightarrow R$ and $g : R \rightarrow R$ are continuous with a, a' bounded, and

$$C(t, s) = C(t - s) = -e^{-(t-s+3)^2} \text{ for } t \geq s \geq 0.$$

It is clear that $C(t, s)$ is not convex (even not positive). If we choose $k = 4$, then

$$D(t, s) = C_t(t, s) + kC(t, s) = 2(t - s + 1)e^{-(t-s+3)^2}.$$

A straightforward calculation shows that $D(t, s)$ is convex and (3.4) holds. We also see that $C(t, s)$ satisfies (3.12). Thus, if there exist constants $L > 0$ and $\mu > 0$ with $xg(x) \geq 0$ for $|x| \geq L$ and

$$g(x)[kx + C(t, t)g(x)] = g(x)[4x - e^{-9}g(x)] \geq \mu g^2(x)$$

for $|x| \geq L$, then any solution of (3.22) is bounded by Theorem 3.2.

4 A Truncated Equation and Unification

We consider the finite delay equation

$$x(t) = a(t) - \int_{t-h}^t C(t, s)g(x(s))ds \quad (4.1)$$

in which $h > 0$ is a constant, $a : R^+ \rightarrow R$, $D : R^+ \times [-h, \infty) \rightarrow R$, $g : R \rightarrow R$ are all continuous with a, a' bounded. We write

$$D(t, s) := C_t(t, s) + kC(t, s) \quad (4.2)$$

for a positive constant k and assume that $D(t, s)$ is convex:

$$D(t, s) \geq 0, D_s(t, s) \geq 0, D_{st}(t, s) \leq 0, D_t(t, s) \leq 0 \quad (4.3)$$

for $t \geq s \geq -h$ and that

$$C(t, t-h) = 0, C_t(t, t-h) = 0, C(t, t) \geq -B \quad (4.4)$$

for all $t \geq 0$ and a constant $B > 0$, where $C_t(t, t-h)$ is the partial derivative of $C(t, s)$ with respect the first variable for $s = t-h$.

Before we get too far into the work, it is interesting to point out classical forms for C . Let

$$C(t, s) = C(t - s) = (-1)^n(t - s - h)^n, \quad n > 2.$$

Not only does it satisfy (4.4), but it is a convex kernel for $0 \leq s \leq t \leq h$. Moreover, if we let $C(t) = 0$ for $t > h$ then that kernel will satisfy our work in both Sections 2 and 3. In Section 3 something very interesting happens. In the linear case we have

$$x(t) = a(t) - \int_0^t C(t - s)x(s)ds,$$

an equation about which there is a very straightforward theory. However, for $t \geq h$ it becomes

$$x(t) = a(t) - \int_{t-h}^t C(t - s)x(s)ds$$

and that belongs to a class of far more complex structure.

We differentiate (4.1) and take into account (4.4) to form

$$x' + kx = \left[a' + ka - C(t, t)g(x) - \int_{t-h}^t D(t, s)g(x(s))ds \right]. \quad (4.5)$$

Now, we define the Liapunov functional

$$V(t) = 2G(x(t)) + \int_{t-h}^t D_s(t, s) \left(\int_s^t g(x(u))du \right)^2 ds \quad (4.6)$$

for $x \in C([-h, \infty), R)$, where $G(x) = \int_0^x g(s)ds$.

Theorem 4.1. *Suppose that (4.3) and (4.4) hold. If $x(t)$ is a solution of (4.1), then the derivative of V along that solution satisfies*

$$V'(t) \leq 2g(x)[a'(t) + ka(t)] - 2g(x)[kx + C(t, t)g(x)]. \quad (4.7)$$

If there is an $L > 0$ with

$$xg(x) \geq 0 \quad \text{for } |x| \geq L \quad (4.8)$$

and if, in addition, there is a $\mu > 0$ with

$$g(x)[kx + C(t, t)g(x)] \geq \mu g^2(x) \quad (4.9)$$

for $|x| \geq L$, then there is an $M > 0$ with

$$V'(t) \leq -|g(x)| + M \quad (4.10)$$

for all $t \geq 0$.

Proof. We first observe that if x is a solution of (4.1), then x is also a solution of (4.5). Now let x be a solution (4.1) and $V(t)$ be defined in (4.6). We then find that

$$\begin{aligned} V'(t) &= 2g(x)x'(t) + \int_{t-h}^t D_{st}(t, s) \left(\int_s^t g(x(u))du \right)^2 ds \\ &\quad - D_s(t, t-h) \left(\int_{t-h}^t g(x(u))du \right)^2 + 2g(x(t)) \int_{t-h}^t D_s(t, s) \int_s^t g(x(u))duds. \end{aligned}$$

Integration of the last term by parts and use of (4.4) yield

$$\begin{aligned} &2g(x(t)) \int_{t-h}^t D_s(t, s) \int_s^t g(x(u))duds \\ &= 2g(x(t)) \left[D(t, s) \int_s^t g(x(u))du \Big|_{s=t-h}^{s=t} + \int_{t-h}^t D(t, s)g(x(s))ds \right] \\ &= 2g(x(t)) \int_{t-h}^t D(t, s)g(x(s))ds. \end{aligned} \tag{4.11}$$

Since $D_{st}(t, s) \leq 0$ and $D_s(t, t-h) \geq 0$, the middle two terms of V' are not positive, and if we use (4.5) and (4.11), we obtain

$$\begin{aligned} V'(t) &\leq 2g(x) \left[-kx + (a' + ka) - C(t, t)g(x) - \int_{t-h}^t D(t, s)g(x(s))ds \right] \\ &\quad + 2g(x) \int_{t-h}^t D(t, s)g(x(s))ds \\ &= 2g(x)[a'(t) + ka(t)] - 2g(x)[kx + C(t, t)g(x)] \end{aligned}$$

verifying (4.7).

Now we assume that (4.8) and (4.9) hold. We may choose $N > 1$ so large that $\mu(N-1) > B$, where B and μ are defined in (4.4) and (4.9), respectively. If $|x| \geq L$, then $xg(x) \geq 0$, and by (4.9), we obtain for $|x(t)| \geq L$ that

$$\begin{aligned} V'(t) &\leq 2|g(x)|[|a'| + k|a|] - 2g(x)[kx + C(t, t)g(x)] \\ &= 2|g(x)|[|a'| + k|a|] - 2k\frac{1}{N}xg(x) \\ &\quad - 2\left(1 - \frac{1}{N}\right)g(x)[kx + C(t, t)g(x)] - 2\frac{1}{N}C(t, t)g^2(x) \\ &\leq 2|g(x)|[|a'| + k|a|] - 2k\frac{1}{N}xg(x) \end{aligned}$$

$$\begin{aligned}
& -2 \left(1 - \frac{1}{N}\right) \mu g^2(x) + 2 \frac{1}{N} B g^2(x) \\
& \leq -2 |g(x)| \left[\frac{1}{N} k |x| - (|a'(t)| + k |a(t)|) \right].
\end{aligned}$$

We may assume that

$$L \geq N \left[\sup_{t \geq 0} (|a'(t)| + k |a(t)|) + 1 \right] / k.$$

Thus, if $|x(t)| \geq L$, then $V'(t) \leq -|g(x(t))|$. Since $-C(t, t) \leq B$, it is clear that $V'(t) \leq M$ for $0 \leq |x(t)| \leq L$, where the constant $M > 0$ is a function of L , and hence,

$$V'(t) \leq -|g(x(t))| + M$$

for all $t \geq 0$. This completes the proof.

To establish boundedness of solutions, we assume that there is a $B_1 > 0$ with

$$D(t, t) \leq B_1 \quad \text{and} \quad \int_{t-h}^t |C_s(t, s)| ds \leq B_1 \quad (4.12)$$

for $t \geq 0$. We then see that

$$\int_{t-h}^t D_s(t, s) ds = D(t, t) \leq B_1 \quad \text{and} \quad |C(t, t)| \leq \int_{t-h}^t |C_s(t, s)| ds \leq B_1.$$

Theorem 4.2. *If (4.3)-(4.4), (4.8)-(4.9), and (4.12) hold, then any solution of (4.1) is bounded.*

Proof. Let x be a solution of (4.1) and $V(t)$ be defined in (4.6). Then $V(t)$ is bounded below and satisfies (4.10). We now show that $V(t)$ is bounded above. If $V(t)$ is unbounded, then there exists a sequence $\{t_n\} \uparrow \infty$ with $V(t_n) \rightarrow \infty$ as $n \rightarrow \infty$ and

$$V(t_n) \geq V(s) \quad \text{for} \quad 0 \leq s \leq t_n.$$

It then follows from (4.10) that

$$0 \leq V(t_n) - V(s) \leq - \int_s^{t_n} |g(x(u))| du + M(t_n - s).$$

This implies that

$$\int_s^{t_n} |g(x(u))| du \leq M(t_n - s) \quad (4.13)$$

and, in particular, that

$$\int_s^{t_n} |g(x(u))| du \leq M(t_n - s) \leq hM \quad (4.14)$$

for all $t_n - h \leq s \leq t_n$. Applying (4.14) to (4.1), We see that

$$\begin{aligned} (x(t_n) - a(t_n))^2 &= \left(\int_{t_n-h}^{t_n} C(t_n, s) g(x(s)) ds \right)^2 \\ &= \left(-C(t_n, s) \int_s^{t_n} g(x(s)) ds \Big|_{s=t_n-h}^{s=t_n} + \int_{t_n-h}^{t_n} C_s(t_n, s) \int_s^{t_n} g(x(u)) du ds \right)^2 \\ &= \left(\int_{t_n-h}^{t_n} C_s(t_n, s) \int_s^{t_n} g(x(u)) du ds \right)^2 \\ &\leq \left(hM \int_{t_n-h}^{t_n} |C_s(t_n, s)| ds \right)^2 \leq h^2 B_1^2 M^2. \end{aligned}$$

This implies that

$$|x(t_n)| \leq \sup_{s \geq 0} |a(s)| + hB_1 M := B_2 \quad (4.15)$$

and that $|G(x(t_n))| \leq B_3$ for a $B_3 > 0$. We now arrive at

$$\begin{aligned} V(t_n) &= 2G(x(t_n)) + \int_{t_n-h}^{t_n} D_s(t_n, s) \left(\int_s^{t_n} g(x(u)) du \right)^2 ds \\ &\leq 2G(x(t_n)) + h^2 M^2 \int_{t_n-h}^{t_n} D_s(t_n, s) ds \\ &\leq 2B_3 + B_1 h^2 M^2 := B_4, \end{aligned} \quad (4.16)$$

a contradiction. Thus, $V(t)$ is bounded. In fact, we have

$$2G(x(t)) \leq V(t) \leq \max\{V(0), B_4\}$$

and hence

$$|V(t)| \leq K \text{ for all } t \geq 0 \quad (4.17)$$

where $K := 2|\eta| + 2|G(x(0))| + B_4$ with $\eta = \inf\{G(u) : u \in R\}$.

We now integrate (4.10) from s to t and use (4.17) to obtain

$$\int_s^t |g(x(u))| du \leq V(s) - V(t) + M(t - s) \leq 2K + M(t - s)$$

and hence

$$\int_s^t |g(x(u))| du \leq 2K + hM \quad \text{for } t_n - h \leq s \leq t_n. \quad (4.18)$$

Applying (4.18) to (4.1) we find that

$$\begin{aligned} |x(t)| &\leq |a(t)| + \left| \int_{t-h}^t C(t, s)g(x(s))ds \right| \\ &\leq |a(t)| + \left| C(t, t-h) \int_{t-h}^t g(x(s))ds + \int_{t-h}^t C_s(t, s) \int_s^t g(x(u))duds \right| \\ &\leq |a(t)| + \int_{t-h}^t |C_s(t, s)|[2K + hM]ds \\ &\leq \sup_{s \geq 0} |a(s)| + B_1(2K + hM). \end{aligned} \quad (4.19)$$

This implies that x is bounded. The proof is complete.

We now consider the existence of periodic solutions of (4.1). We assume that $a : R \rightarrow R$, $C : R \times R \rightarrow R$, and $g : R \rightarrow R$ are continuous and that there is a $T > 0$ with

$$a(t+T) = a(t), \quad C(t+T, s+T) = C(t, s) \quad (4.20)$$

for all $t \geq s$. If (4.20) holds, then $C(t, t)$ and $\int_{t-h}^t |C_s(t, s)|ds$ are T -periodic, and so there are B and B_1 with

$$C(t, t) \geq -B, \quad \int_{t-h}^t |C_s(t, s)|ds \leq B_1;$$

then (4.4) and (4.12) are satisfied. We define a companion of (4.1) by

$$x(t) = \lambda \left[a(t) - \int_{t-h}^t C(t, s)g(x(s))ds \right], \quad 0 \leq \lambda \leq 1 \quad (4.21)$$

for $t \in R$ and form a differential equation

$$x' + kx = \lambda \left[a' + ka - C(t, t)g(x) - \int_{t-h}^t D(t, s)g(x(s))ds \right]. \quad (4.22)$$

To obtain an *a priori* bound for all T -periodic solutions of (4.21), we define

$$V_1(t) = 2G(x(t)) + \lambda \int_{t-h}^t D_s(t, s) \left(\int_s^t g(x(u))du \right)^2 ds \quad (4.23)$$

for $t \in R$ and $x \in (P_T, \|\cdot\|)$.

Theorem 4.3. *If (4.3)-(4.4), (4.8)-(4.9), and (4.20) hold for $t \geq s$, then (4.1) has a T -periodic solution.*

Proof. Let x be a T -periodic solution of (4.21) and $V_1(t)$ be defined in (4.23). Then we have

$$V_1'(t) \leq -|g(x)| + M \quad (4.24)$$

for $t \geq 0$ and for an $M > 0$ independent of x and λ . Since $V_1(t)$ is T -periodic, $V_1(t)$ has a global maximum at $q \in [0, T]$, and hence, at $t_n = q + nT$. We then have

$$0 \leq V_1(t_n) - V_1(s) \leq - \int_s^{t_n} |g(x(u))| du + M(t_n - s)$$

for all $s \leq t_n$. An argument similar to that of (4.13)-(4.16) shows that $V_1(t_n) \leq B_4$ with B_4 defined just after (4.16). Observing that

$$V_1(0) \leq V_1(t_n) \leq B_4,$$

we see that $|V_1(t)| \leq K$ with $K = 2|\eta| + B_4$, where $\eta = \inf\{G(u) : u \in R\}$. We then follow the argument in (4.19) to arrive at

$$|x(t)| < \sup_{s \geq 0} |a(s)| + B_1(2K + hM) + 1 := B^* \quad (4.25)$$

for all $t \in R$. This implies that $\|x\| < B^*$ whenever x is a T -periodic solution of (4.21) for $0 \leq \lambda \leq 1$.

Define a mapping F on P_T by

$$F(\phi)(t) = a(t) - \int_{t-h}^t C(t, s)g(\phi(s))ds \quad (4.26)$$

for each $\phi \in P_T$. It is clear that $F(\phi) \in P_T$. We will show that F is continuous on P_T and is compact on each bounded subset of P_T . If $\tilde{\phi}, \phi \in P_T$, then

$$\begin{aligned} |F(\phi)(t) - F(\tilde{\phi})(t)| &= \left| \int_{t-h}^t C(t, s)g(\phi(s))ds - \int_{t-h}^t C(t, s)g(\tilde{\phi}(s))ds \right| \quad (4.27) \\ &= \left| \int_{t-h}^t C_s(t, s) \left(\int_s^t g(\phi(v))dv - \int_s^t g(\tilde{\phi}(v))dv \right) ds \right|. \end{aligned}$$

Since g is uniformly continuous on $\{x \in R : |x| \leq \|\tilde{\phi}\| + 1\}$, then for any $\varepsilon > 0$, there exists $0 < \delta < 1$ such that $\|\phi - \tilde{\phi}\| < \delta$ implies $|g(\phi(s)) - g(\tilde{\phi}(s))| < \varepsilon$ for all

$s \in [0, T]$. It follows from (4.27) that $\|F(\phi) - F(\tilde{\phi})\| \leq hB_1\varepsilon$. Thus, F is continuous on P_T .

We now show that F is compact on each bounded subset of P_T . Let $\eta > 0$ and define

$$\Gamma = \{F(\phi) : \phi \in P_T, \|\phi\| \leq \eta\}. \quad (4.28)$$

Observe that

$$\begin{aligned} \frac{d}{dt}F(\phi)(t) &= a'(t) - C(t, t)g(\phi(t)) - \int_{t-h}^t C_t(t, s)g(\phi(s))ds \\ &= a'(t) - C(t, t)g(\phi(t)) - \int_{t-h}^t D(t, s)g(\phi(s))ds + k \int_{t-h}^t C(t, s)g(\phi(s))ds \\ &= a'(t) - C(t, t)g(\phi(t)) - \int_{t-h}^t D(t, s)g(\phi(s))ds + k \int_{t-h}^t C_s(t, s) \int_s^t g(\phi(v))dv ds \end{aligned}$$

and that

$$\begin{aligned} \left| \frac{d}{dt}F(\phi)(t) \right| &\leq \|a'\| + g^* \sup_{0 \leq t \leq T} \left[|C(t, t)| + \int_{t-h}^t D(t, s)ds + k \int_{t-h}^t |C_s(t, s)|(t-s)ds \right] \\ &\leq \|a'\| + g^* \left[\sup_{0 \leq t \leq T} |C(t, t)| + hB_1 + hkB_1 \right] \end{aligned}$$

where $g^* = \sup\{|g(u)| : |u| \leq \eta\}$; thus, Γ is equi-continuous. The uniform boundedness of Γ follows from the inequality

$$|F(\phi)(t)| \leq \|a\| + \int_{t-h}^t |C_s(t, s)| \int_s^t |g(\phi(v))|dv \leq \|a\| + hB_1g^*$$

for all $\phi \in \Gamma$. So, by the Ascoli-Arzela theorem, Γ lies in a compact subset of P_T . By Schaefer's theorem, we see F has a fixed point which is a T -periodic solution of (4.1). The proof is complete.

We now give two examples which show a connection between this section and Sections 2 and 3.

Example 1. Consider the scalar equation

$$x(t) = a(t) - \int_{-\infty}^t C(t-s)g(x(s))ds \quad (4.29)$$

where

$$\begin{aligned} C(t) &= (-1)^n(t-h)^n, \quad 0 \leq t \leq h, \quad n = 3, 4, \dots \\ &= 0, \quad t \geq h \end{aligned} \quad (4.30)$$

for some $h > 0$. It is readily verified that C'' is continuous for $0 \leq t < \infty$ and that $C(t-s)$ is convex. Moreover, using two changes of variable we find that

$$\begin{aligned} x(t) &= a(t) - \int_0^\infty C(u)g(x(t-u))du \\ &= a(t) - \int_0^h C(u)g(x(t-u))du \\ &= a(t) - \int_{t-h}^t C(t-s)g(x(s))ds \end{aligned} \quad (4.31)$$

and $C(h) = C'(h) = C''(h) = 0$. All of the work in Sections 2 and 4 hold for this equation.

Example 2. Consider

$$x(t) = a(t) - \int_0^t C(t-s)g(x(s))ds \quad (4.32)$$

with solution ϕ on $[0, h]$ where C satisfies (4.30). Then for $t \geq h$ we have

$$x(t) = a(t) - \int_{t-h}^t C(t-s)g(x(s))ds, \quad t \geq h, \quad (4.33)$$

with initial function ϕ on $[0, h]$.

Theorem 4.4 *Let $\frac{dg(x)}{dx}$ be continuous, let $x(t)$ be the unique solution of (4.32), and let $y(t)$ be any continuous solution of (4.33). Suppose that there is an $L > 0$ with $\frac{dg(x)}{dx} \geq L$. Then $z(t) := x(t) - y(t) \in L^2[h, \infty)$. If, in addition, there is an $M > 0$ with $\frac{dg(x)}{dx} \leq M$, then $z(t) \rightarrow 0$ as $t \rightarrow \infty$.*

Proof. We have for $t \geq h$ that

$$z(t) = - \int_{t-h}^t C(t-s)[g(x(s)) - g(y(s))]ds \quad (4.34)$$

$$= - \int_{t-h}^t C(t-s) \frac{dg(\xi(s))}{dx} z(s)ds \quad (4.35)$$

where $\xi(s)$ is between $x(s)$ and $y(s)$. Define a Liapunov functional by

$$V(t) = \int_{t-h}^t C_s(t-s) \left(\int_s^t [g(x(u)) - g(y(u))]du \right)^2 ds \quad (4.36)$$

with derivative satisfying

$$V'(t) \leq -2[g(x(t)) - g(y(t))][x(t) - y(t)] = -2 \frac{dg(\xi(t))}{dx} z^2(t) \leq -2Lz^2(t). \quad (4.37)$$

This yields the first conclusion. With the last assumption, note that

$$|z(t)| \leq M \int_{t-h}^t |C(t-s)||z(s)|ds \quad (4.38)$$

$$\leq M \sqrt{\int_{t-h}^t C^2(t-s)ds} \sqrt{\int_{t-h}^t z^2(s)ds} \quad (4.39)$$

$$\leq M \sqrt{h^{2n+1} \int_{t-h}^t z^2(s)ds} \quad (4.40)$$

and

$$\int_{t-h}^t z^2(s)ds \rightarrow 0 \text{ as } t \rightarrow \infty. \quad (4.41)$$

Under the conditions here, with C defined by (4.30) we see that the solutions of the equations in Sections 2, 3, and 4 all converge to the same function both pointwise and in L^2 .

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