Stability Theorems for Nonautonomous Functional Differential Equations by Liapunov Functionals

T.A. Burton*
Department of Mathematics
Southern Illinois University
Carbondale, IL 62901–4408

László Hatvani†
Bolyai Institute
Aradi vértanúk tere 1
H–6720 szeged
Hungary

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1 Introduction

We consider the system

$$x'(t) = F(t, x_t) \tag{1}$$

where x_t is that segment of $x(s)$ on $[t-h, t]$ shifted to $[-h, 0]$, where $h > 0$ is a fixed constant, and where x' denotes the right-hand derivative. The following notation will be used.

For $x \in R^n$, $|x| = \max |x_i|$. For $h > 0$, C denotes the space of continuous functions mapping $[-h, 0]$ into R^n, and for $\phi \in C$, $\|\phi\| = \sup_{-h \le \theta \le 0} |\phi(\theta)|$. Also, C_H denotes the set of $\phi \in C$ with $\|\phi\| < H$. If x is a continuous function of u defined for $-h \le u < A$, with $A > 0$, and if t is a fixed number satisfying $0 \le t < A$, then x_t denotes the restriction of x to $[t-h, t]$ so that x_t is an element of C defined by $x_t(\theta) = x(t+\theta)$ for $-h \le \theta \le 0$. We denote by $x(t_0, \phi)$ a solution of (1) with initial condition $\phi \in C$ where $x_{t_0}(t_0, \phi) = \phi$ and we

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denote by $x(t, t_0, \phi)$ the value of $x(t_0, \phi)$ at t.

It is supposed that $F : R_+ \times C_H \to R^n$ is continuous and takes bounded sets into bounded sets; here, $H > 0$ or $H = \infty$. It is known ([4], [10], or [17]) that for each $t_0 \in R_+$ and each $\phi \in C_H$ there is at least one solution $x(t_0, \phi)$ defined on an interval $[t_0, t_0 + \alpha)$ and, if there is an $H_1 < H$ with $|x(t, t_0, \phi)| \leq H_1$, then $\alpha = \infty$. Here, $R_+ = [0, \infty)$.

The object of this paper is to give conditions on Liapunov functionals to ensure stability and boundedness of solutions of (1). This is, of course, an old problem and there are many known results and applications. In fact, it was a survey of those results and particularly the applications which inspired this investigation.

A Liapunov functional is a continuous function $V(t, \phi)$ from $R_+ \times C_H \to R_+$ whose derivative along a solution of (1) satisfies some specific relation. The derivative of a Liapunov functional $V(t, \phi)$ along a solution $x(t)$ of (1) may be defined in several equivalent ways. If V is differentiable, the natural derivative is obtained using the chain rule. But we may take

$$V'_{(1)}(t, \phi) = \limsup_{\delta \to 0^+} \{V(t + \delta, x_{t+\delta}(\cdot, t, \phi)) - V(t, \phi)\}/\delta.$$

Important and informative discussions of the various derivatives are found in Yoshizawa [17; pp. 186–188] and Driver [7].

DEFINITION 1. Let $F(t, 0) = 0$.

(a) The zero solution of (1) is said to be *stable* if for each $\epsilon > 0$ and $t_0 \geq 0$ there is a $\delta > 0$ such that $[\phi \in C_\delta,\ t \geq t_0]$ imply that $|x(t, t_0, \phi)| < \epsilon$.

(b) The zero solution is *uniformly stable* (U.S.) if it is stable and if δ is independent of t_0.

(c) The zero solution is *asymptotically stable* (A.S.) if it is stable and if for each $t_0 \geq 0$ there is a $\delta > 0$ such that $\phi \in C_\delta$ implies that $x(t, t_0, \phi) \to 0$ as $t \to \infty$.

(d) The zero solution is *uniformly asymptotically stable* (U.A.S.) if it is U.S. and if there is an $\eta > 0$ and for each $\gamma > 0$ there exists $T > 0$ such that $[t_0 \in R_+,\ \phi \in C_\eta,\ t \geq t_0 + T]$ imply that $|x(t, t_0, \phi)| < \gamma$.

The earliest results on Liapunov's direct method for such equations tended to be patterned on those for ordinary differential equations with the norm in R^n replaced by the supremum norm in the function space C. Results so stated were easy to prove and some of them could be reversed yielding converse theorems. But examples could almost never be found satisfying the stated conditions. In particular, if we let W_i denote continuous functions from $R_+ \to R_+$, $W_i(0) = 0$, and $W_i(r)$ strictly increasing (called *wedges*) then the results often asked for

(i) $W_1(\|\phi\|) \leq V(t, \phi)$, $V(t, 0) = 0$ and

(ii) $V'_{(1)}(t, x_t) \leq -W_2(\|x_t\|)$.

(See Krasovskii [13], Halanay [9], and El'sgol'ts [8] for discussions.) Examples were readily constructed with

(i)′ $W_1(|\phi(0)|) \leq V(t, \phi)$, $V(t, 0) = 0$ and

(ii)″ $V'_{(1)}(t, x_t) \leq -W_2(|x(t)|)$.

These conditions frequently suffice to prove good stability results and, at this writing, the following is a summary of the commonly accepted results.

THEOREM 1. Let $V : R_+ \times C_H \to R_+$ be continuous:

(a) If (i) $W_1(|\phi(0)|) \leq V(t, \phi)$, $V(t, 0) = 0$, and
(ii) $V'_{(1)}(t, x_t) \leq 0$
then $x = 0$ is stable.

(b) If (i) $W_1(|\phi(0)|) \leq V(t, \phi) \leq W_2(\|\phi\|)$ and
(ii) $V'_{(1)}(t, x_t) \leq 0$
then $x = 0$ is U.S.

(c) If (i) $W_1(|\phi(0)|) \leq V(t, \phi) \leq W_2(\|\phi\|)$ and
(ii) $V'_{(1)}(t, x_t) \leq -W_3(\|x_t\|)$
then $x = 0$ is U.A.S.

(d) If (i) $W_1(|\phi(0)|) \leq V(t, \phi) \leq W_2(\|\phi\|)$
(ii) $V'_{(1)}(t, x_t) \leq -W_3(|x_t|)$
(iii) $F(t, \phi)$ is bounded for ϕ bounded
then $x = 0$ is U.A.S.

(e) If $|||\phi|||$ denotes the L^2-norm and

(i) $W_1(|\phi(0)|) \leq V(t,\phi) \leq W_2(|\phi(0)|) + W_3(|||\phi|||)$ and

(ii) $V'_{(1)}(t,x_t) \leq -W_4(|x(t)|)$

then $x = 0$ is U.A.S.

While parts (a), (b), and (c) can be reversed, (c) has not proved to be useful. All parts of the theorem can be traced to some degree to Krasovskii [13] (cf. Driver [7]). Part (e) was proved in [3]. The condition of F bounded for ϕ bounded is a straightforward generalization of the classical Marachkoff result (cf. [1]).

While almost all investigators have insisted on asking $V(t,\phi) \leq W_2(\|\phi\|)$, virtually all known examples use a simple variant of $V(t,\phi) \leq W_2(|\phi(0)|) + W_3(|||\phi|||)$ which is far more flexible, as (e) indicates and as we show throughout this paper.

While almost all investigators have, quite correctly, dropped the requirement that $V'_{(1)}(t,x_t) \leq -W(||x_t||)$ since it is almost never realized in applications, we know of none who have utilized the condition $V'_{(1)}(t,x_t) \leq -W(|||x_t|||)$, a condition present in a great number of standard examples. We will show in this paper that such conditions greatly facilitate proofs of strong stability.

While investigators have discovered many interesting results for ordinary differential equations in recent years from the relation $V'(t,x) \leq -\eta(t)W(|x|)$ where η is integrally positive, few attempts at such results have been made for functional differential equations. One such discussion is found in Yoshizawa [18]. We consider the relation $V'_{(1)}(t,x_t) \leq -\eta(t)W(|x_t|_r)$ where $|\cdot|_r$ is some norm and present variants of integral positivity which are very fruitful in establishing stability and boundedness when $\eta(t)$ is zero on intervals of length less than h.

We emphasize that it has been very difficult for investigators to obtain asymptotic stability without asking $F(t,\phi)$ bounded for ϕ bounded (Busenberg and Cooke [6] recently made an advance here; see our Example D). This difficulty vanishes when we use a norm on C in V', and such norms are common in applications.

2 A Survey of Examples

In this section we look at several well known examples noting especially the properties which investigators have used and pointing out additional properties which could have been used to great advantage. These examples will serve to motivate some of the results in the following sections.

In a great many examples there appears in the Liapunov functional a term like

$$\int_{-h}^{0}\int_{t+s}^{t} D(u, x_u)du\,ds$$

where D is some non-negative functional. There then appears in the derivative terms like

$$-\eta_1(t)W(|x(t)|) - \eta_2(t)\int_{t-h}^{t} D(s, x_s)ds$$

where η_1 and η_2 are non-negative functions. Almost always the second term is discarded (cf. [10; pp. 55–57] and [17; pp. 206–210]); investigators obtain qualitative results when $\eta_1(t)$ is integrally positive. But we note here that the second term may be much more useful. Results may be obtained when η_2 vanishes on sets of length less than h.

EXAMPLE A. Consider the scalar equation

$$x'(t) = b(t)x(t-h) \tag{A1}$$

where $b : [-h, \infty) \to [-1, 0]$ is continuous,

$$-2 + \int_{t-h}^{t} |b(u)|du + h \leq 0, \tag{A2}$$

$$b(t+h) = b(t), \quad \int_{t-h}^{t} |b(u)|du > 0, \tag{A3}$$

and

$$\int_{t-h}^{t} \big[1 - |b(s)|\big]ds > 0. \tag{A4}$$

These conditions imply that all solutions tend to zero as $t \to \infty$.

To see this, define

$$V(t,x_t) = \left[x(t) + \int_{t-h}^{t} b(u)x(u)du\right]^2 + \int_{-h}^{0}\int_{t+s}^{t} |b(u)|x^2(u)du\,ds$$

and obtain

$$V'(t,x_t) \le |b(t)|\left[-2 + \int_{t-h}^{t} |b(u)|du + h\right]x^2 + (|b(t)|-1)\int_{t-h}^{t} |b(s)|x^2(s)ds.$$

If, for example, $b(t)$ is a classical square wave periodic function which is zero and then -1 (smoothed), then the term

$$V'(t,x_t) \le -\alpha|b(t)|x^2(t), \quad \alpha > 0,$$

is without value in the classical theory. But $1-|b(t)|$ has the same square wave character and, because of (A4), our Theorem 2 shows that $\int_{t-h}^{t} |b(u)|x^2(u)du \to 0$, hence, that

$$\left|\int_{t-h}^{t} b(u)x(u)du\right| \le \left[\int_{t-h}^{t} |b(u)|du \int_{t-h}^{t} |b(u)|x^2(u)du\right]^{1/2}$$

tends to zero as $t \to \infty$. Thus, $V(t,x_t) - x^2(t)$ tends to zero and $V(t,x_t)$ tends to a constant c; but $c = 0$, otherwise, $\int_{t-h}^{t} |b(u)|x^2(u)du$ cannot tend to zero.

As a comparison, we note that Krasovskii's theorem on asymptotic stability of the zero solution of

$$x'(t) = g(x(t-h(t)),t)$$

for $0 \le h(t) \le h$ (see [13; p. 174]), when applied to (A1), requires that $b(t) \le -h-\gamma$ for $t \ge 0$ and for some $\gamma > 0$; this condition is not met by the square wave function.

After Theorem 2 we will deal with the nonlinear generalization of (A1) with arbitrary (non-periodic) b.

EXAMPLE B. Hale [11; pp. 120–123] discusses an example of Levin and Nohel concerning a circulating fuel nuclear reactor and a viscoelastic model in the form of a scalar equation

$$x'(t) = -\int_{t-h}^{t} a(t-u)g(x(u))du, \tag{B1}$$

where $G(x)=\int\limits_0^x g(s)ds \to \infty$ as $|x|\to\infty$, $a(h)=0$, $a(t)\geq 0$, $a(t)\not\equiv 0$, $a'(t)\leq 0$, $a''(t)\geq 0$ for $t\in[0,h]$; also, the functions a'' and g are continuous, while g has only isolated zeros.

Applying sophisticated theory of limit sets, Hale proved the following nice theorem which gives a complete description of asymptotic behavior of solutions of (B1).

THEOREM B1 (Hale [11; p. 122]). (i) If there is an s such that $a''(s)>0$, then, for any $\phi\in C$, the ω-limit set $\omega(\phi)$ of the orbit through ϕ is an equilibrium point of (B1), i.e., a zero of g.

(ii) If $a''(s)\equiv 0$, $a\not\equiv 0$, then for any $\phi\in C$ the ω-limit set $\omega(\phi)$ of the orbit through ϕ is a single periodic orbit of period h generated by a solution of the equation

$$x''+a(0)g(x)=0.$$

In his proof, Hale defines the functional

$$V(\phi)=G(\phi(0))-(1/2)\int_{-h}^{0}a'(-u)\left[\int_u^0 g(\phi(s))ds\right]^2du$$

with derivative along a solution of (B1) being

$$V'(\phi)=(1/2)a'(h)\left[\int_{-h}^{0}g(\phi(u))du\right]^2-(1/2)\int_{-h}^{0}a''(-u)\left[\int_u^0 g(\phi(s))ds\right]^2du.$$

He then uses the invariance principle.

By contrast, our results here hold in the non-autonomous case. From Theorem 4 it follows that for every solution we have

$$\lim_{t\to\infty}\int_{-h}^{0}\int_{-h}^{0}a''(-u)\left[\int_u^0 g(x(t+v+s))ds\right]^2du\,dv=0$$

and if $a'(h)\neq 0$, then

$$\lim_{t\to\infty}\int_{-h}^{0}\int_{-h}^{0}g(x(t+v+s))ds\,dv=0.$$

Using these facts one can obtain the assertions in a simple way. Since autonomous theory is not required, one may generalize Theorem B1 to non-autonomous equations. We illustrate this after Theorem 4 by a result on the equation

$$x'(t) = -\alpha(t)\int_{t-h}^{t} a(t-s)g(x(t+s))ds$$

where $\alpha : R_+ \to R_+$.

EXAMPLE C. Krasovskii [13] considered the nonlinear second order equation

$$x''(t) + \phi\big(x'(t), t) + f(x(t-h(t))\big) = 0 \tag{C1}$$

where $f : R \to R$ has a continuous derivative, while $\phi : R \times R_+ \to R_+$ and $h : R_+ \to R_+$ are continuous and periodic in t and $0 \le h(t) \le h$ and h is constant.

Using the notation $y(t) = x'(t)$ and $f^*(x) = df(x)/dx$, one can rewrite (C1) as the equivalent system

$$\begin{cases} x'(t) = y(t) \\ y'(t) = -\phi(y(t), t) - f(x(t)) + \displaystyle\int\limits_{-h(t)}^{0} f^*(x(t+s))y(t+s)ds. \end{cases} \tag{C2}$$

Krasovskii defined the Liapunov functional

$$V(x_t, y_t) = 2\int_0^x f(s)ds + y^2(t) + v^2\int_{-h}^{0}\int_u^0 y^2(t+s)ds\,du \tag{C3}$$

where $0 < v^2$ is constant, and he obtained

$$V'(x_t, y_t) \le -\gamma\Big[y^2(t) + \int_{t-h}^{t} y^2(u)du\Big], \tag{C4}$$

where γ is a positive constant, under the conditions

$$\phi(y, t)/y \ge b > 0 \quad \text{for} \quad t \ge 0 \quad \text{and} \quad y \ne 0 \tag{C5}$$

and

$$f(x)/x \ge a > 0, \quad |f^*(x)| \le L \tag{C6}$$

with a, b, and L constant.

Using techniques which he developed for autonomous and periodic systems (later called the invariance principle), he proved that if (C5) and (C6) are satisfied and $b > Lh$ then the zero solution of (C1) is asymptotically stable.

It is known (see, e.g., [12]) that in the non-retarded case ($h(t) \equiv 0$), conditions (C5) and (C6) imply asymptotic stability without any extra condition.

It is interesting to note that the appearance of a delay can neutralize the effect of friction (the term $\phi(x'(t), t)$) and can destablilize the equilibrium $x = 0$. Somolinos [16] investigated the sunflower equation

$$x''(t) + bx'(t) + L \sin x(t - h) = 0 \tag{C7}$$

in which b, L, and h are positive constants, $b \geq L$, and $bh \geq 1$. This is a special case of (C1) for small x. Using properties of the characteristic equation he proved several interesting results including the following one.

THEOREM C. Let ξ be the root of $\sigma^2/Lh^2 = \cos\sigma$ in $(0, \pi/2)$ and define $b_0 = Lh(\sin\xi)/\xi$ (so that $(2/\pi)Lh < b_0 < Lh$).

(i) If $b > b_0$, then the zero solution of (C7) is asymptotically stable.

(ii) If $b < b_0$, then the zero solution of (C7) is unstable.

(iii) For fixed L and h equation (C7) has a Hopf bifurcation at $b = b_0$; consequently, if $b < b_0$, then (C7) has a nontrivial periodic solution.

Now the following questions arise: What can be said about the stability properties of the zero solution of (C1) when $\phi(y, t)$ and $h(t)$ are not periodic in t and when $\phi(y, t)$ is possibly an unbounded function of t? What conditions guarantee asymptotic stability? Is the condition

$$\inf\{\phi(y, t)/y : y \neq 0\} \geq b > Lh$$

necessary for the asymptotic stability? (This is of interest even if ϕ and h are periodic in t.)

The invariance principle and method of characteristic equations cannot be used to solve these problems. The difficulties are caused by the fact that we have to replace the third term in the Liapunov functional (C3) by

$$\int_{-h}^{0}\int_{u}^{0} \nu^2(t+s)y^2(t+s)ds\,du.$$

This changes (C4) to a more complicated form

$$V'(t,x_t,y_t) \le -\eta_1(t)y^2(t) - \eta_2(t)\int_{t-h}^{t} \xi(t+u)y^2(u)du;$$

but our theorems make it possible to handle this inequality, as we show after Theorem 4.

EXAMPLE D. Busenberg and Cooke [6] address the problem of improving Theorem 1(d) and motivate their work with the scalar equation

$$x'(t) = -a(t)x(t) + b(t)x(t-h) \tag{D1}$$

with $a : R_+ \to R_+$ and $b : R_+ \to R$ continuous. They proved uniform asymptotic stability for the zero solution under the following conditions: for each $\eta > 0$ there exists $\tau > 0$ such that

$$\int_{t}^{t+\tau} |b(s)|ds < \eta \quad \text{for} \quad t \ge 0 \tag{D2}$$

(so that

$$\int_{t-h}^{t} |b(u+h)|du \le B \tag{D3}$$

for some $B > 0$), and for some $p, q > 0$ the inequality

$$2a(t) - p|b(t)| - |b(t+h)|/p \ge q \tag{D4}$$

holds for $t \ge 0$.

They define the functional

$$V(t,\phi) = p\phi^2(0) + \int_{-h}^{0} K(t+u)\phi^2(u)du$$

where $K(t) = |b(t+h)|$ and they show that

$$V'(t,\phi) = [K(t) - 2pa(t)]\phi^2(0) + 2pb(t)\phi(0)\phi(-h) - K(t-h)\phi^2(-h),$$

from which they conclude that V' is a negative definite quadratic form in $\phi(0)$ and $\phi(-h)$; however, they use only the pair

$$\begin{aligned} W_1(|\phi(0)|) \le V(t,\phi) \le W_2(\|\phi\|) \\ V'(t,\phi) \le -W_3(|\phi(0)|) \end{aligned} \tag{D5}$$

instead of the pair (D4) and

$$V'(t,x_t) \le -\eta_4(t)W_4(|x(t)|) - \eta_5(t)W_5(|x'(t)|) \tag{D6}$$

for appropriate η_4 and η_5. Applying our Theorem 6 to this pair yields the following: the zero solution of (D1) is U.A.S. if (D3) is satisfied and for some $\beta > 1$ we have

$$\eta(t) = a(t) - \beta|b(t+h)| \ge 0, \quad \lim_{s\to\infty} \int_t^{t+s} \eta(u)du = \infty \tag{D7}$$

uniformly with respect to $t \in R_+$.

On the other hand, using another Liapunov functional V_1 satisfying the pair

$$V_1(t,\phi) \le 2\phi^2(0) + c\int_{-h}^0 |b(t+h+u)|\phi^2(u)du$$

and

$$V'_{(1)}(t,\phi) \le -\gamma(t)\phi^2(0) - \eta(t)\int_{-h}^0 |b(t+h+u)|\phi^2(u)du$$

(with c constant) we get conditions which guarantee asymptotic stability for the zero solution. These conditions work, for example, when $b(t+h) \equiv -a(t)$ in which case neither (D4) nor (D7) hold.

3 Integral Positivity

Liapunov's direct method centers around a relation

$$W_1(|x(t)|) \le V(t,x_t)$$

in which one drives $|x(t)|$ to zero by driving $V(t, x_t)$ to zero. This involves relating the derivative of V to an upper bound on V. Typically, one has a relation

$$V'_{(1)}(t, x_t) \leq -W_2(|x(t)|)$$

so that

$$V(t, x_t) \leq V(t_0, x_{t_0}) - \int_{t_0}^{t} W_2(|x(s)|)ds.$$

The classical theory then relies almost exclusively on asking that the solution move slowly so that unless $|x(t)| \to 0$, then $V(t, x_t) \to -\infty$, a contradiction. But that idea is crude, inefficient, and dreadfully wasteful of the tools at hand. In this section we explore three techniques which seem natural for the types of examples which investigators have constructed to this point.

Virtually always investigators feel that the delay in (1) complicates the problem of stability and that something more is needed than is required in equations without a delay. For example, Theorem 1(d) is true without asking that $F(t, \phi)$ be bounded for ϕ bounded when there is no delay. We point out here that frequently the delay simplifies the problem and that many theorems are true when $h > 0$, but they become false when $h = 0$.

In this section we are interested in relations including $V'_{(1)}(t, x_t) \leq -\eta(t)W(|||x_t|||)$. The basic concept needed here is a generalization of integral positivity which has found significant application in ordinary differential equations (cf. Hatvani [12], Matrosov [14], Murakami [15], and Yoshizawa [19]). We point out that this concept can be considerably weakened for delay equations, while retaining the same results.

DEFINITION 2. A measurable function $\eta : R_+ \to R_+$ is said to be *integrally positive with parameter* $\delta > 0$ (IP(δ)) if whenever $I = \bigcup_{m=1}^{\infty} [\alpha_m, \beta_m]$ with $\alpha_m < \beta_m < \alpha_{m+1}$ and $\beta_m - \alpha_m \geq \delta$ $(m = 1, 2, \dots)$, then $\int_I \eta(t)dt = \infty$.

If a function η is integrally positive for every $\delta > 0$ then it is called integrally positive (IP).

For example, the function

$$\eta_1(t) = |\cos t| - \cos^2 t$$

is IP, while

$$\eta_2(t) = |\cos t| - \cos t$$

is IP(δ) whenever $\delta > \pi$, but it is not IP.

It can be seen that a measurable function $\eta : R_+ \to R_+$ is IP(δ) if and only if

$$\liminf_{t\to\infty} \int_t^{t+\delta} \eta(s)ds > 0.$$

The following lemma points out that regardless of how fast a function may change in magnitude, its integral from $t-h$ to t does not change rapidly.

LEMMA 1. For a continuous function $x : R_+ \to R^n$ and a continuous functional $D : R_+ \times C \to R_+$ define the function $g(t) = \int\limits_{t-h}^{t} D(s, x_s)ds$. Given $\epsilon > 0$ and $0 < h_1 < h$ let $\delta = \epsilon(h-h_1)/(2h-h_1)$. If $g(t_1) \geq \epsilon$ for some $t_1 > 2h$, then there is a closed interval $[a, b]$ of length h_1 containing t_1 in which $g(t) \geq \delta : b - a = h_1$, $t_1 \in [a, b]$, $g(t) \geq \delta$ for all $t \in [a, b]$.

PROOF. Let $\gamma = h - h_1$ and

$$N = \begin{cases} h/\gamma & \text{if } h/\gamma \text{ is an integer} \\ [h/\gamma] + 1 & \text{otherwise} \end{cases}$$

where $[h/\gamma]$ is the greatest integer function. Construct the intervals $I_1 = [t_1 - h, t_1 - h + \gamma]$, $I_2 = [t_1 - h + \gamma, t_1 - h + 2\gamma]$, ... , $I_N = [t_1 - h + (N-1)\gamma, t_1]$. Then for some i we have $\int\limits_{I_i} D(t, x_t)dt \geq \epsilon/N$ and we denote the right end-point of I_i by t_2. Then $I_i \subset [t_2+h_1-h, t_2+h_1]$ and so $g(t) \geq \epsilon/N$ for $t \in [t_2, t_2+h_1] =: [a, b]$. Since $N \leq (h/\gamma)+1$ this completes the proof.

THEOREM 2. (A) Let $D, V : R_+ \times C_H \to R_+$ be continuous and suppose there are continuous functions $\eta_1 : R_+ \to R_+$ and $B : R_+ \to [0, \infty)$ with B nondecreasing. Suppose also that for every $\epsilon > 0$ there is an $h_1 \in (0, h)$ such that the function

$$t \mapsto \eta_1(t) W_1 \big[W_2^{-1}(\epsilon / B(t+h))(h - h_1)/(2h - h_1) \big]$$

is IP(h_1) and

(i) $V'_{(1)}(t, x_t) \le -\eta_1(t) W_1 \big[\int_{t-h}^{t} D(s, x_s) ds \big].$

Then for every solution $x(t)$ of (1) satisfying $\|x_t\| < H$ on $[t_0, \infty)$ there is the relation $\lim_{t\to\infty} \left\{ B(t) W_2 \big[\int_{t-h}^{t} D(s, x_s) ds \big] \right\} = 0$. In particular ($B(t) \equiv 1$), if $\eta_1 \in$ IP(h_1) for $h_1 \in (0, h)$ and (i) is satisfied, then every solution of (1) satisfying $\|x_t\| < H$ satisfies $\lim_{t\to\infty} \int_{t-h}^{t} D(s, x_s) ds = 0$.

(B) In addition to the conditions in (A), suppose there is a continuous function $\eta_5 : R_+ \to R_+$ such that $\int_0^\infty \eta_5(t) dt = \infty$,

(ii) $W_3(|x(t)|) \le V(t, x_t) \le W_4(|x(t)|) + B(t) W_2 \big[\int_{t-h}^{t} D(s, x_s) ds \big]$

and

(iii) $V'_{(1)}(t, x_t) \le -\eta_5(t) W_5(|x(t)|)$.

Then $x = 0$ is A.S.

(C) Let the conditions in (A) hold and, in addition, suppose

(iv) there exists a function $V^* : R^n \to R_+$ such that if $x : [t_0 - h, \infty) \to R^n$ is a solution of (1) with $\lim_{t\to\infty} \left\{ B(t) W_2 \big[\int_{t-h}^{t} D(s, x_s) ds \big] \right\} = 0$, then

$$\lim_{t\to\infty} \big[V(t, x_t) - V^*(x(t)) \big] = 0.$$

Then for every solution $x(t)$ satisfying $\|x_t\| < H$ on $[t_0, \infty)$, the finite limit $\lim_{t\to\infty} V^*(x(t))$ exists. Moreover, if there is a continuous $\eta_6 : R_+ \to R_+$ with $\int_0^\infty \eta_6(t) dt = \infty$.

(v) $W_3(|\phi(0)|) \le V(t, \phi)$, $V(t, 0) = 0$,

and (vi) $V'_{(1)}(t, x_t) \le -\eta_6(t) W_6\big(V^*(x(t))\big)$,

then $x = 0$ is A.S.

PROOF. Let $x(t)$ be a solution of (1) satisfying $\|x_t\| < H$ on $[t_0, \infty)$ and define $g(t) = \int_{t-h}^{t} D(s, x_s)ds$. If $B(t)W_2(g(t)) \nrightarrow 0$ as $t \to \infty$, then there is an $\epsilon > 0$ and a sequence $\{t_i\}$ with $t_{i+1} > t_i + 2h$ for which $B(t_i)W_2(g(t_i)) \geq \epsilon$. Applying Lemma 1 to the function g we obtain a sequence $\{\bar{t}_i\}$ with $t_i - h \leq \bar{t}_i \leq t_i$ and $g(t) \geq W_2^{-1}[\epsilon/B(t_i)](h - h_1)/(2h - h_1)$ for $\bar{t}_i \leq t \leq \bar{t}_i + h_1$. Thus, for $t \geq \bar{t}_k + h_1$ we have

$$0 \leq V(t, x_t) \leq V(t_0, x_{t_0}) - \sum_{i=1}^{k} \int_{\bar{t}_i}^{\bar{t}_i + h_1} \eta_1(s) W_1\big[W_2^{-1}(\epsilon/B(s+h))(h - h_1)/(2h - h_1)\big] ds$$

because $B(t)$ is nondecreasing. Thus, by the assumption on the integrand, we see that $V(t, x_t) \to -\infty$ as $t \to \infty$, a contradiction. This proves (A).

To prove (B), we first note that $x = 0$ is stable by Theorem 1(a). Let $x(t)$ be a solution with $\|x_t\| < H$ for $t \geq t_0$. Then (iii) implies that $\liminf_{t\to\infty} |x(t)| = 0$. Using (ii) and (A) we have $\limsup_{t\to\infty} W_3(|x(t)|) \leq \lim_{t\to\infty} V(t, x_t) \leq \liminf_{t\to\infty} W_4(|x(t)|) + \lim_{t\to\infty} B(t)W_2\big[\int_{t-h}^{t} D(s, x_s)ds\big] = 0$, completing the proof of (B).

To prove (C), we begin by showing that every solution $x(t)$ with $\|x_t\| < H$ on $[t_0, \infty)$ satisfies the limit relation

$$\lim_{t\to\infty} V^*(x(t)) = \lim_{t\to\infty} V(t, x_t) =: V_0.$$

By Theorem 2(A) $\lim_{t\to\infty} \left\{ B(t)W_2\big(\int_{t-h}^{t} D(s, x_s)ds\big) \right\} = 0$, so, in consequence of (iv), for every $\epsilon > 0$ there is a $T_1(\epsilon)$ such that $|V^*(x(t)) - V(t, x_t)| < \epsilon/2$ for all $t \geqq T_1(\epsilon)$. On the other hand, there exists a $T_2(\epsilon)$ such that $|V(t, x_t) - V_0| < \epsilon/2$ for all $t \geqq T_2(\epsilon)$. Consequently, $|V^*(x(t)) - V_0| \leqq |V^*(x(t)) - V(t, x_t)| + |V(t, x_t) - V_0| < \epsilon$ for all $t \geqq \max\{T_1(\epsilon), T_2(\epsilon)\}$, which completes the proof of existence of the limit.

If (v) and (vi) hold, then $x = 0$ is stable by Theorem 1(a). By virtue of (v), to show A.S. it is enough to prove that for any solution $x(t)$ with $\|x_{t_0}\|$ small enough then $V_0 = \lim_{t\to\infty} V(t, x_t) = 0$. Suppose that $V_0 > 0$. Then $V^*(x(t)) > V_0/2$ for $t \geq \bar{T}$, some $\bar{T}$, and by

(vi) we have

$$V(t,x_t) \le V(\bar{T},x_{\bar{T}}) - W_6(V_0/2)\int_{\bar{T}}^{t}\eta_6(s)ds \to -\infty,$$

a contradiction. This completes the proof.

REMARK 1. Inequalities (iii) and (vi) can be replaced by the following conditions, respectively: for each continuous function $u : R_+ \to R^n$, then

(iii′) $\liminf_{t\to\infty} |u(t)| > 0$ implies that

$$\limsup_{t\to\infty}\left[B(t)W_2\left[\int_{t-h}^{t} D(s,u_s)ds\right]\right] > 0$$

and

(vi′) $\lim_{t\to\infty} V^*(u(t)) > 0$ implies that

$$\limsup_{t\to\infty}\left[B(t)W_2\left[\int_{t-h}^{t} D(s,u_s)ds\right]\right] > 0.$$

Indeed, from the proof of Theorem 2(B) it can be seen that it is enough to show that $\liminf_{t\to\infty} |x(t)| = 0$. By (iii′), if this is not true then we get a contradiction to the assertion of Theorem 2(A).

As regards (vi′), the single role of (vi) in the proof of Theorem 2(B) was to guarantee that $V_0 = \lim_{t\to\infty} V^*(x(t)) = 0$. By condition (vi′), the assumption $V_0 > 0$ is in contradiction to Theorem 2(A); thus (vi) can be replaced by (vi′).

EXAMPLE 1. Consider the scalar equation

$$x'(t) = -a(t)x(t) + b(t)\int_{t-h}^{t} x(u)du$$

in which $a, b : R_+ \to R$ are continuous with

(i) $-a(t) + \alpha \int_{t}^{t+h} |b(s)|ds \le 0$ for some $\alpha > 1$ and

(ii) $|b(t)|/B(t+h)$ is IP(h_1) for some $0 < h_1 < h$, where $B(t) := \max_{0\le s\le t} \int_{s}^{s+h} |b(u)|du$.

Then $x = 0$ is A.S.

PROOF. Define

$$V(t, \phi) = |\phi(0)| + \alpha \int_{-h}^{0} \int_{s}^{0} |b(t + u - s)|\, |\phi(u)| du\, ds$$

so that

$$\begin{aligned} |\phi(0)| \le V(t, \phi) &\le |\phi(0)| + \alpha \int_{-h}^{0} \int_{t+u}^{t+u+h} |b(v)| dv |\phi(u)| du \\ &\le |\phi(0)| + \alpha B(t) \int_{-h}^{0} |\phi(u)| du. \end{aligned}$$

Also

$$\begin{aligned} V'(t, x_t) &\le -a(t)|x(t)| + |b(t)| \int_{t-h}^{t} |x(u)| du \\ &\qquad + \alpha \int_{-h}^{0} |b(t - s)|\, |x(t)| ds - \alpha \int_{-h}^{0} |b(t)|\, |x(t + s)| ds \\ &= \left[- a(t) + \alpha \int_{t}^{t+h} |b(s)| ds \right] |x(t)| + |b(t)|(1 - \alpha) \int_{t-h}^{t} |x(u)| du \\ &\le |b(t)|(1 - \alpha) \int_{t-h}^{t} |x(u)| du. \end{aligned}$$

Therefore, the conditions of Theorem 2(B) and (iii′) in Remark 1 are satisfied, and $x = 0$ is A.S.

REMARK 2. The following properties in this example are noteworthy.

(i) The conditions depend on the size of h.

(ii) The function $F(t, \phi)$ need not be bounded for ϕ bounded, and $\int_{t}^{t+h} |b(s)| ds$ can also be unbounded.

(iii) We do not have V' dependent on $|x(t)|$, but rather on its integral; in particular, given $b(t) \in \mathrm{IP}(h_1)$, define $a(t) = \alpha \int_{t}^{t+h} |b(s)| ds$.

(iv) The stability actually comes from $a(t)$ through the relation (i) even though the derivative of V ultimately centers on $b(t)$.

EXAMPLE D revisited. Consider again the scalar equation (D1). If for $t \in R_+$ we have

(i) $a(t) \geq b(t+h)$,

(ii) $\gamma(t) := [-a(t) + b(t+h)]\big[2 - \int_{t-h}^{t} |b(u+h)|du\big] + \alpha h|b(t+h)| \leq 0$, and

(iii) $\eta(t) := \alpha - a(t) + b(t+h)$ is IP(h_1) for some $\alpha > 0$ and $h_1 \in (0, h)$,

then every solution tends to a finite limit as $t \to \infty$.

If, in addition, one of the conditions

(iv_1) $\limsup_{t\to\infty} \int_t^{t+h} |b(s)|ds > 0$,

(iv_2) $\eta(t) \int_{t}^{t+h} |b(s)|ds \notin L^1[0, \infty)$, or

(iv_3) $\int_{0}^{\infty} \gamma(t)dt = -\infty$,

then the zero solution is A.S.

PROOF. We can write equation (D1) as

$$x'(t) = [-a(t) + b(t+h)]x(t) - (d/dt)\int_{t-h}^{t} b(u+h)x(u)du$$

and define

$$V(t,\phi) = \left[\phi(0) + \int_{-h}^{0} b(t+u+h)\phi(u)du\right]^2 + \alpha\int_{-h}^{0}\int_{s}^{0} |b(t+u+h)|\phi^2(u)du\,ds$$

so that

$$\begin{aligned}
V'(t,x_t) = 2&\left[x(t) + \int_{t-h}^{t} b(u+h)x(u)du\right][-a(t) + b(t+h)]x(t) \\
&+ \alpha\int_{-h}^{0} |b(t+h)|x^2(t)ds - \alpha\int_{-h}^{0} |b(t+s+h)|x^2(t+s)ds \\
\leq &\left\{[-a(t) + b(t+h)] + |-a(t) + b(t+h)|\int_{t-h}^{t} |b(u+h)|du + \alpha h|b(t+h)|\right\}x^2(t) \\
&+ (|-a(t) + b(t+h)| - \alpha)\int_{t-h}^{t} |b(s+h)|x^2(s)ds \\
= &\,\gamma(t)x^2(t) - \eta(t)\int_{t-h}^{t} |b(s+h)|x^2(s)ds.
\end{aligned}$$

Equation (D1) is linear so every solution can be continued for all $t \geq t_0$. Conditions (i)–(iii) yield the conditions of Theorem 2(A); thus, we have

$$\lim_{t\to\infty} \int_{t-h}^{t} |b(u+h)| x^2(u)du = 0$$

for each solution $x(t)$. Conditions (i)–(ii) imply $\int\limits_{t-h}^{t} |b(u+h)|du \leq 2$ for all $t \geq 0$, whence we get the estimate

$$\begin{aligned} V(t,\phi) &\leq 2\phi^2(0) + \left[2\int_{t-h}^{t} |b(u+h)|du + \alpha h\right] \int_{-h}^{0} |b(t+h+u)|\phi^2(u)du \\ &\leq 2\phi^2(0) + (4+\alpha h)\int_{-h}^{0} |b(t+h+u)|\phi^2(u)du. \end{aligned}$$

Obviously, the limit $V^*(\phi(0))$ exists and $V^*(\phi^2(0)$, in the notation of Theorem 2(C). By the first part of Theorem 2(C) we know that $\lim_{t\to\infty} x^2(t)$ exists, so $\lim_{t\to\infty} x(t) = x_0$ exists also. This means that every solution is bounded; therefore, the zero solution is stable (cf. [11; p. 162]). We still must prove that $x_0 = 0$.

As $V'(t, x_t) \leq \gamma(t)x^2(t)$, if condition (iv$_3$) holds then all conditions of Theorem 2(C) hold and $x_0 = 0$.

If (iv$_1$) holds, then we use Remark 1. For $\lim_{t\to\infty} V^*(u(t)) = \lim_{t\to\infty} u^2(t) = u_0^2 > 0$, then

$$\begin{aligned} \limsup_{t\to\infty} \int_{t-h}^{t} D(s,u_s)ds &= \limsup_{t\to\infty} \int_{t-h}^{t} |b(s+h)|u^2(s)ds \\ &\geq (u_0^2/2) \limsup_{t\to\infty} \int_{t}^{t+h} |b(s)|ds > 0, \end{aligned}$$

so $x = 0$ is A.S. by Theorem 2(C) and Remark 1.

If (iv$_2$) is satsified and if there is a solution with $x_0 \neq 0$, then

$$V(t,x_t) \leq V(t_0, x_{t_0}) - \int_{t_0}^{t} \eta(s) \int_{s-h}^{s} |b(p+h)|\,[x_0^2/2]dp\,ds \to -\infty \quad \text{as} \quad t \to \infty$$

and this is a contradiction.

EXAMPLE A revisited. Consider the scalar equation

$$x'(t) = b(t)f(x(t-h)) \tag{A5}$$

which is a nonlinear generalization of (A1). Here, $b : R_+ \to [-\alpha, 0]$ $(0 < \alpha = \text{constant})$, $f : R \to R$ are continuous. Suppose that there is a constant $c > 0$ such that

(i) $xf(x) \geq 0$ and $|f(x)| \leq c|x|$ for all x,

(ii) $|b(t)| \leq \alpha \leq (2/ch) - (1/h) \int_t^{t+h} |b(s)|ds$ for $t \in R_+$, and

(iii) $\alpha - |b(t)|$ is IP(h_1) for some $h_1 \in (0, h)$.

Then every solution tends to a limit as $t \to \infty$. If in addition,

(iv) $xf(x) > 0$ for $x \neq 0$ and

(v) $\int_0^\infty |b(t)|dt = \infty$,

then $x = 0$ is A.S.

PROOF. Define

$$V(t, x_t) = \left(x(t) + \int_{t-h}^{t} b(s+h)f(x(s))ds\right)^2 + \alpha \int_{-h}^{0}\int_{t+u}^{t} |b(s+h)|f^2(x(s))ds\,du$$

so that

$$\begin{aligned} V'(t, x_t) &= 2\left(x(t) + \int_{t-h}^{t} b(s+h)f(x(s))ds\right) b(t+h)f(x(t)) \\ &\quad + \alpha \int_{-h}^{0} |b(t+h)|f^2(x(t))du - \alpha \int_{-h}^{0} |b(t+u+h)|f^2(x(t+u))du \\ &\leq 2b(t+h)x(t)f(x(t)) + |b(t+h)|\left[\alpha h + \int_t^{t+h} |b(s)|ds\right] f^2(x(t)) \\ &\quad + (|b(t+h)| - \alpha)\int_{t-h}^{t} |b(s+h)|f^2(x(s))ds. \end{aligned}$$

Taking into account (i) we obtain

$$\begin{aligned} V'(t, x_t) &\leq |b(t+h)| \left[-2 + c\left(\alpha h + \int_t^{t+h} |b(s)|ds\right)\right] x(t)f(x(t)) \\ &\quad + (|b(t+h)| - \alpha)\int_{t-h}^{t} |b(s+h)|f^2(x(s))ds. \end{aligned}$$

By Theorem 2(A), (ii) and (iii) imply that

$$\lim_{t\to\infty} \int_{t-h}^{t} |b(s+h)|f^2(x(s))ds = 0 \tag{A6}$$

for each solution $x(t)$ which is defined on $[0, \infty)$. On the other hand,

$$V(t, x_t) \leq 2x^2(t) + \left[2 \int_t^{t+h} |b(s)| ds + \alpha h \right] \int_{t-h}^{t} |b(s+h)| f^2(x(s)) ds$$
$$\leq 2x^2(t) + K \int_{t-h}^{t} |b(s+h)| f^2(x(s)) ds.$$

Obviously, denoting $V^*(x) = x^2$, by (A6) we have

$$\lim_{t \to \infty} \left[V(t, x_t) - V^*(x(t)) \right] = 0.$$

In order to apply Theorem 2(C) we have to show that every solution is defined on $[t_0, \infty)$. If there is a solution $x : [t_0 - h, T) \to R$ which is noncontinuable, then $\limsup_{t \to T^-} |x(t)| = \infty$. Since we can verify from the equation that $x'(t)$ is bounded on $[t - h, T)$, such behavior is impossible.

By the first assertion of Theorem 2(C) every solution $x(t)$ has a finite limit k_x, proving our claim. To prove A.S., we first show stability. Since

$$x(t+h) - x(t) = \int_t^{t+h} x'(s) ds = \int_t^{t+h} b(s) f(s-h) ds = \int_{t-h}^{t} b(u+h) f(x(u)) du,$$

we have

$$V(t, x_t) = x^2(t+h) + \alpha \int_{-h}^{0} \int_{t+u}^{t} |b(s+h)| f^2(x(s)) ds\, du \geq x^2(t+h).$$

For $\epsilon > 0$ and $t_0 \in R_+$ given numbers, choose $\delta(\epsilon, t_0) > 0$ so that

$$\sup_{\|\phi\| < \delta} V(t_0, \phi) < \epsilon^2, \qquad \sup_{(\|\phi\| < \delta,\, t_0 \leq t \leq t_0 + h)} \|x(t, t_0, \phi)\| < \epsilon.$$

Let $x(t) = x(t, t_0, \phi)$ be a solution with $\|\phi\| < \delta$, and suppose that $|x(T)| = \epsilon$ for some $T > t_0 + h$. Then

$$\epsilon^2 = x^2(T) \leq V(T - h, x_{T-h}) \leq V(t_0, \phi) < \epsilon^2,$$

a contradiction. This proves $x = 0$ is stable.

We now show that the limit k_x of the arbitrary solution x is zero. If

$$\limsup_{t\to\infty} \int_t^{t+h} |b(s)|ds > 0 \quad \text{and if} \quad k_x \neq 0,$$

then by (A6) we have

$$\begin{aligned} 0 &= \lim_{t\to\infty} \int_{t-h}^{t} |b(s+h)| f^2(x(s))ds \\ &\geq \min\{f^2(r) : |k_x|/2 \leq |r| \leq 2|k_x|\} \limsup_{t\to\infty} \int_{t-h}^{t} |b(s+h)|ds > 0. \end{aligned}$$

Suppose that $\lim\limits_{t\to\infty} \int\limits_t^{t+h} |b(s)|ds = 0$. From (ii) and (v) we obtain $c_0 = 2 - \alpha ch > 0$. Therefore, if $k_x \neq 0$, then

$$\begin{aligned} V'(t, x_t) &\leq -(c_0/2)|b(t+h)|x(t)f(x(t)) \\ &\leq -(c_0/2)|b(t+h)|(|k_x|/2) \min\{f(r) : |k_x|/2 \leq |r| \leq 2|k_x|\} \end{aligned}$$

for large t. Then by (iv) and (v) we see that $V(t, x_t) \to -\infty$, a contradiction.

REMARK 3. Krasovskii [13; p. 174] investigated the scalar equation

$$x' = g(x(t - h(t)), t)$$

where $0 \leq h(t) \leq h$, h constant, with $g(x,t)$, $\partial g(x,t)/\partial x$ continuous and

$$|\partial g(x,t)/\partial x| < L = \text{ const.} \quad (t \geq 0,\ x \in R). \tag{A7}$$

Using a Liapunov-Razumikhin method he showed that if

$$[g(x,t)/x] + L^2 h(t) \leq -\gamma \quad (t \geq 0,\ x \in R) \tag{A8}$$

is satisfied with an arbitrarily small positive constant γ, then $x = 0$ is A.S. We show that for (A5), Krasovskii's (A7) and (A8) imply our (i)–(v), but the converse is false.

Indeed, suppose that (A7) and (A8) hold for (A5). Since in this example $\partial g(x,t)/\partial x = b(t)f'(x)$, (A7) asks $|f'(x)| \leq K$ with K a suitable constant, and (A8) can be rewritten as

$$[b(t)f(x)/x] + \alpha^2 K^2 h < -\gamma \quad (x \neq 0,\ t \in R_+).$$

Then by Lagrange's mean value theorem, our conditions (i) and (iv) are satisfied, namely, $c = K$ can be chosen. Obviously, condition (v) holds also.

Since $|b(t)| \leq \alpha$ we have

$$(2/ch) - (1/h)\int_t^{t+h} |b(s)|ds - \alpha = 2\alpha[1/(\alpha ch) - 1].$$

If $c\alpha h < 1$, then (ii) is satisfied, and we can assume without loss of generality that (iii) holds as well; otherwise, α must be replaced by $\alpha' > 0$ so that $c\alpha' h < 1$. Consequently, it is enough to prove that (A7) and (A8) imply $c\alpha h < 1$.

Suppose that (A7) and (A8) hold with $c\alpha h \geq 1$. Then

$$-c\alpha \leq b(t)f(x)/x < -\gamma - \alpha^2 c^2 h < -\gamma - \alpha c < -\alpha c,$$

a contradiction.

On the other hand, our conditions allow $b(t)$ to vanish (even on intervals), while (A8) can not be satisfied for such a function.

We now propose a theorem on A.S. and U.A.S. in the case when the function η in the inequality

$$V'_{(1)}(t, x_t) \leq -\eta(t)W\left[\int_{t-h}^{t} D(s, x_s)ds\right]$$

is not integrally positive. We know from the theory of ordinary differential equations that the zero solution of

$$x'(t) = -(1/(t+1))x$$

is A.S. (using $V(x) = x^2$), even though $\eta(t) = 1/(t+1)$ is not integrally positive.

Suppose now that

$$W_1(|\phi(0)|) \leq V(t, \phi) \leq W_2(|\phi(0)|) + W_3(|||\phi|||) \leq W_4(\|\phi\|)$$

and $V'_{(1)}(t,x_t) \le 0$. If $x(t)$ is a bounded solution which does not tend to zero then there is an $\epsilon > 0$ and a sequence $\{t_i\} \uparrow \infty$ with $|x(t_i)| \ge \epsilon$. One uses a variety of devices to show that $|x(t)| \ge \epsilon/2$ on intervals $t_i \le t \le t_i + \gamma$ for some $\gamma > 0$. If one has

$$V'_{(1)}(t,x_t) \le -\eta(t)W(\epsilon/2) \quad \text{on} \quad [t_i, t_i+\gamma]$$

then one hopes to show that $\sum_{i=1}^{\infty} \int_{t_i}^{t_i+\gamma} \eta(s)ds = \infty$. This is impossible to do when $\eta(t) = 1/(t+1)$ unless one can also show that $t_{i+1} - t_i$ is bounded. But because $V(t,\phi) \le W_4(\|\phi\|)$ and $V'(t,x_t) \le 0$ one can frequently show that $t_{i+1} - t_i$ is bounded.

DEFINITION 3. Let $\eta : R_+ \to R_+$ be measurable.

(a) The function η is *weakly integrally positive* with parameters $\delta > 0$ and $\Delta > 0$ (WIP(δ,Δ)) if whenever $\{t_i\}$ and $\{\delta_i\}$ satisfy $t_i + \delta_i < t_{i+1} \le t_i + \delta_i + \Delta$ with $\delta_i \ge \delta$, then

$$\sum_{i=1}^{\infty} \int_{t_i}^{t_i+\delta_i} \eta(t)dt = \infty.$$

(b) The function η is *uniformly weakly integrally positive* with parameters $\delta > 0$ and $\Delta > 0$ (UWIP(δ,Δ)) if (a) holds and if for every $M > 0$ there exists $Q > 0$ such that for all $S > Q$ and for all $\{t_i\}$ and $\{\delta_i\}$ satisfying (a), then

$$\int_{[t_1,t_1+S]\cap I} \eta(t)dt > M \quad \text{where} \quad I = \bigcup_{i=1}^{\infty} [t_i, t_i+\delta_i].$$

REMARK 4. If η is IP(δ), then it is UWIP(δ,Δ) for all $\Delta > 0$. The converse is false.

Indeed, let η be IP(δ), i.e.,

$$\liminf_{t\to\infty} \int_t^{t+\delta} \eta(s)ds = 2c > 0.$$

Then there is a T such that $t \ge T$ implies that $\int_t^{t+\delta} \eta(s)ds \ge c$. Choosing $Q(M) = T + (M/c)(\delta+\Delta)$, we get the first assertion. To obtain the second part it suffices to consider the function

$$\eta(t) = \begin{cases} 0 & \text{if } k^2 \le t \le k^2+\delta,\ k = 1,2,\dots, \\ 1 & \text{otherwise.} \end{cases}$$

It is UWIP(δ, Δ) for all $\Delta > 0$, but it is not IP(δ).

THEOREM 3. Let η be WIP$(h_1, 4h)$ with $0 < h_1 < h$ and suppose that the continuous functionals $D, V : R_+ \times C_H \to R_+$ satisfy:

(i) $D(t, \phi) \leq W_1(\|\phi\|)$;

(ii) for some $K \in (0, H)$ there exists a wedge W_2 such that $[t \in R_+, u : [-2h, 0] \to R^n$ is continuous, $|u(s)| \leqq K$ for $s \in [-2h, 0]$ imply

$$W_2\Big(\inf\{|u(r)| : -h \leqq r \leqq 0\}\Big) \leqq \int_{-h}^{0} D(t+s, u_s)ds;$$

(iii) for every continuous function $\alpha : [-2h, \infty) \to R^n$ the inequality

$$W_3(|\alpha(t)|) \leq V(t, \alpha_t) \leq W_4(|\alpha(t)|) + W_5\left[\int_{t-h}^{t} D(s, \alpha_s)ds\right]$$

is satisfied for all $t \in R_+$;

(iv) $V'_{(1)}(t, x_t) \leq -\eta(t) W_6\big[\int\limits_{t-h}^{t} D(s, x_s)ds\big]$.

Then $x = 0$ is A.S. If, in addition, η is UWIP$(h_1, 4h)$, then $x = 0$ is U.A.S.

PROOF. By conditions (i) and (iii) there is a W_7 with $V(t, \phi) \leq W_7(\|\phi\|)$; therefore, it follows from Theorem 1(b) that $x = 0$ is U.S. For the $K \in (0, H)$ let $\delta = \delta(K) > 0$ be that of U.S. Let $\gamma > 0$, $t_0 \geq 0$, $\phi \in C_\delta$ be given. We will find $T = T(\gamma)$ such that $t \geq t_0 + T$ implies that $|x(t, t_0, \phi)| < \gamma$. Find $\xi > 0$ and $W_4(\xi) + W_5(W_2(\xi)) < W_3(\gamma)$. Consider the intervals

$$I_j = [t_0 + jh, t_0 + (j+1)h], \quad j = 0, 1, 2, \dots .$$

Suppose that for each j there is a $t_j \in I_j$ with $g(t_j) \geq W_2(\xi)$, Where $g(t) = \int\limits_{t-h}^{t} D(s, x_s)ds$. Use Lemma 1 to find $\bar{\delta} > 0$ and $\bar{t}_j \in I_{j-1} \cup I_j$ such that $g(t) \geq \bar{\delta}$ for $\bar{t}_j \leq t \leq \bar{t}_j + h_1$. Then the intervals $[\bar{t}_{3j}, \bar{t}_{3j} + h_1]$ are disjoint and the sequence $\{\bar{t}_{3j}\}$ satisfies the conditions of η being WIP$(h_1, 4h)$. Thus, there is an $N = N(t_0)$ for which

$$W_6(\bar{\delta}) \sum_{i=1}^{N} \int_{\bar{t}_{3i}}^{\bar{t}_{3i}+h_1} \eta(s)ds > W_7(\delta).$$

Moreover, if η is UWIP$(h_1, 4h)$, then N is independent of t_0. An integration of $V'_{(1)}(t, x_t) \leq -\eta(t)W_6\big[\int\limits_{t-h}^{t} D(s, x_s)ds\big]$ from t_0 to $t > \bar{t}_{2N} + h_1$ will contradict $V(t, x_t) \geq 0$.

Thus, there is an I_j with $j \leq 2N$ for which $\int\limits_{t-h}^{t} D(s, x_s)ds < W_2(\xi)$ for all $t \in I_j$. Because of (ii) this means that there is a $t_* \in I_j$ with $|x(t_*)| < \xi$. Hence for $t \geq t_*$ we have

$$W_3(|x(t)|) \leq V(t, x_t) \leq V(t_*, x_{t_*}) \leq W_4(\xi) + W_5(W_2(\xi)) < W_3(\gamma).$$

Thus, $T = 2Nh$ and the proof is complete.

EXAMPLE 2. Consider the nonlinear scalar equation

$$x'(t) + \int_{t-h}^{t} b(s)g(x(s))ds = 0$$

with continuous functions $b : R_+ \to R$, $g : R \to R$, and suppose that the following conditions are satisfied:

(i) $b(t) \geq 0$ and $\int\limits_{t-h}^{t} b(s)ds \geq \beta$
for some constant $\beta > 0$ and all $t \geq 0$;

(ii) $xg(x) > 0$ and $|g(x)| \leq c|x|$ for all $x \in R$ and some $c > 0$, c constant;

(iii) there is a $k > 0$ such that for $t \in R_+$ then

$$h^2 b(t) \leq k \leq (2/c) - h\int_{t-h}^{t} b(s)ds;$$

(iv) the function $k - h^2 b(t)$ is WIP$(h_1, 4h)$ for some $h_1 \in (0, h)$.

Then $x = 0$ is A.S. If, in addition, $k - h^2 b(t)$ is UWIP$(h_1, 4h)$, then $x = 0$ is U.A.S.

PROOF. We apply Theorem 3 with the functional

$$V(t, x_t) = \left[x(t) - \int_{-h}^{0}\int_{t+s}^{t} b(u)g(x(u))du\,ds\right]^2 + k\int_{-h}^{0}\int_{t+s}^{t} b(u)g^2(x(u))du\,ds.$$

We first show there is a W_3 with $V(t, x_t) \geq W_3(|x(t)|)$. Let

$$I = \left|\int_{-h}^{0}\int_{t+s}^{t} b(u)g(x(u))du\,ds\right|.$$

If $I < |x(t)|/2$, then $V(t, x_t) \geq x^2(t)/4$. If $I \geq |x(t)|/2$, then

$$\begin{aligned} x^2(t)/4 \leq I^2 &= \left| \int_{t-h}^{t} b(u)g(x(u))(u-t+h)du \right|^2 \\ &\leq \int_{t-h}^{t} b(u)(u-t+h)du \int_{t-h}^{t} b(u)(u-t+h)g^2(x(u))du \\ &\leq k \int_{-h}^{0} \int_{t+s}^{t} b(u)g^2(x(u))du \leq V(t, x_t). \end{aligned}$$

Thus, $W_3(r) = r^2/4$.

It is also easy to show that

$$V(t, x_t) \leq 2x^2(t) + K \int_{t-h}^{t} b(u)g^2(x(u))du$$

for some $K > 0$.

By using (i) and (ii) the derivative of V can be estimated as follows:

$$\begin{aligned} V'(t, x_t) &= -2hb(t)g(x(t)) \left[x(t) - \int_{-h}^{0} \int_{t+s}^{t} b(u)g(x(u))du\, ds \right] \\ &\quad + hkb(t)g^2(x(t)) - k \int_{-h}^{0} b(t+s)g^2(x(t+s))ds \\ &\leq -\gamma(t)x(t)g(x(t)) - \eta(t) \int_{t-h}^{t} b(u)g^2(x(u))du \end{aligned}$$

where $\eta(t) = k - h^2 b(t) \geq 0$ and

$$\gamma(t) = hb(t) \left[2 - c\left(k + h \int_{t-h}^{t} b(u)du \right) \right] \geq 0.$$

Setting $D(t, x_t) = b(t)g^2(x(t))$ we obtain the assertion from Theorem 3.

It can be observed that the main role in the estimate for V' is played by the term $\eta(t) \int_{t-h}^{t} b(u)g^2(x(u))du$. In the next section we establish a method giving the main role to the other member of V', namely $[\gamma(t)/b(t)]b(t)x(t)g(x(t))$ (see Theorem 5). It is easy to see that in the case of the continuous function $b(t)$ defined by

$$b(t) = \begin{cases} 1/(ch^2) & \text{if } 7ih \leq t \leq (7i+1)h \\ 1/(2ch^2) & \text{if } (7i+2)h \leq t \leq (7i+6)h \\ \text{linear elsewhere} & \end{cases}$$

and by choosing $k = 1/c$, all conditions of Example 2 are met and, consequently, $x = 0$ is U.A.S. On the other hand, in this case $\gamma(t)/b(t) = h\big[1 - ch \int\limits_{t-h}^{t} b(s)ds\big]$ is not positive in measure (see Definition 4), so Theorem 5 can not be applied.

4 Reverse Schwarz Inequality

In this section we concentrate on Liapunov functions satisfying estimates of the type

$$V(t, x_t) \le W_1(|x(t)|) + W_2(|||x_t|||)$$

and

$$V'_{(1)}(t, x_t) \le -\eta(t) W(|x(t)|).$$

Use of these inequalities requires a type of "reverse Schwarz inequality". In particular, if there is a $t_1 \ge 0$, an $\epsilon > 0$, and an $\alpha > 0$ such that $|||x_{t_1}||| \ge \epsilon$ and $\int\limits_{t_1-h}^{t_1} \eta(s)ds \ge \alpha$, then we will need to show that there is a $\beta > 0$ with $\int\limits_{t_1-h}^{t_1} \eta(s)W(|x(s)|)ds \ge \beta$ and β is independent of t_1. That is the problem which motivates the next definition.

DEFINITION 4. A measurable function $\eta : R_+ \to R_+$ is said to be *positive in measure* (PIM) if for every $\epsilon > 0$ there are $T \in R_+$, $\delta > 0$ such that $[t \ge T,\ Q \subset [t-h, t]$ is open, $\mu(Q) \ge \epsilon]$ imply that $\int\limits_{Q} \eta(t)dt \ge \delta$. (Here, $\mu(Q)$ denotes the Lebesgue measure of Q.)

For example, the functions $\eta_1(t) \equiv 1$, $\eta_2(t) = \sin^2 t$, and

$$\eta_3(t) = \begin{cases} 1 & \text{if } n \le t \le (n+1) - 1/n \\ 0 & \text{if } n + 1 - 1/n < t < n + 1 \end{cases}$$

are PIM.

LEMMA 2. Let $K > 0$ be given and suppose that η is PIM. Then for each W_1 and $\alpha > 0$ there are $\beta > 0$ and $T \in R_+$ such that if $f : R_+ \to R$ is measurable, $f^2(s) \le K$ for $s \in R_+$, $t \ge T$, $\int\limits_{t-h}^{t} f^2(s)ds \ge \alpha$, then $\int\limits_{t-h}^{t} \eta(s)W_1(|f(s)|)ds \ge \beta$.

PROOF. For a given f with $f^2(s) < K$ on R_+ for $\alpha > 0$, and for $t > h$ suppose that $\int_{t-h}^{t} f^2(s)ds \geq \alpha$. Choose r such that $0 < r < \alpha/h$ and define

$$Q_r(t) = \{s : t-h \leq s \leq t,\ f^2(s) > r\}.$$

Using the notation $Q_r^c(t) = [t-h, t]\backslash Q_r(t)$ we have

$$\alpha \leq \int_{t-h}^{t} f^2(s)ds = \int_{Q_r(t)} f^2(s)ds + \int_{Q_r^c(t)} f^2(s)ds \leq \mu(Q_r(t)) \cdot K + \mu(Q_r^c(t))r.$$

Consequently

$$\alpha \leq \mu(Q_r(t))K + \big[h - \mu(Q_r(t))\big]r$$

and

$$0 < (\alpha - hr)/(K - r) \leq \mu(Q_r(t)).$$

Now, take the numbers T and $\delta > 0$ belonging to $\epsilon = (\alpha - hr)/(K - r)$ in the sense of the definition of positivity in measure. If $t \geq T$ then

$$\int_{t-h}^{t} \eta(s)W_1(|f(s)|)ds \geq W_1(\sqrt{r}) \int_{Q_r(t)} \eta(s)ds \geq W_1(\sqrt{r})\delta =: \beta > 0,$$

which completes the proof.

THEOREM 4. Let η be PIM and let $D, V : R_+ \times C_H \to R_+$ both be continuous with

(i) $0 \leq V(t, \phi)$ and

(ii) $V'_{(1)}(t, x_t) \leq -\eta(t)W_4(D(t, x_t))$.

If $x(t)$ is any solution of (1) such that $|x(t)| < H$ and $D(t, x_t)$ is bounded on $[t_0, \infty)$, then $\lim_{t\to\infty} \int_{t-h}^{t} D(s, x_s)ds = 0$.

PROOF. Suppose $x(t)$ is such a solution but that $\int_{t-h}^{t} D(s, x_s)ds \not\to 0$. Then there is an $\epsilon > 0$ and $\{t_i\} \uparrow \infty$ with $t_{i+1} > t_i + h$ and $\int_{t_i-h}^{t_i} D(s, x_s)ds \geq \epsilon$. By Lemma 2 we can find a T_*

and a $\beta > 0$ with $\int\limits_{t_i-h}^{t_i} \eta(s)W_4(D(s,x_s))ds \geq \beta$ if $t_i \geq T_*$. This means that $V(t,x_t) \to -\infty$ as $t \to \infty$, a contradiction.

EXAMPLE C revisited. Consider again system (C2), where we are assuming that $f : R \to R$ is continuously differentiable, $|f^*(x)| \leq L$ for all $x \in R$ and $xf(x) > 0$ for $x \neq 0$, $\phi : R \times R_+ \to R_+$, $h : R_+ \to R_+$, ϕ and h are continuous, $0 \leq h(t) \leq h$ for all $t \in R_+$.

For an arbitrary solution of (C2) consider the Liapunov functional

$$V(x_t,y_t) = 2F(x(t)) + y^2(t) + \int_{-h}^{0}\int_{t+u}^{t} \nu^2(s)y^2(s)ds\,du, \tag{C8}$$

where $F(x) = \int\limits_0^x f(s)ds$ and $\nu : R_+ \to (0,\infty)$ is a given measurable function.

Using the notation

$$b(t) = \inf\{\phi(y,t)/y : y \neq 0\},$$

we obtain

$$\begin{aligned} V'(t,x_t,y_t) &= -2\phi(y(t),t)y(t) + 2y(t)\int_{-h(t)}^{0} f^*(x(t+s))y(t+s)ds \\ &\quad + \int_{-h}^{0}[\nu^2(t)y^2(t) - \nu^2(t+u)y^2(t+u)]du \\ &\leq -\int_{-h}^{0}[(2b(t)/h - \nu^2(t))y^2(t) - 2L|y(t)|\,|y(t+u)| + \nu^2(t+u)y^2(t+u)]du. \end{aligned}$$

Taking into account the identity

$$\mu u^2 - 2Luv + \nu^2v^2 = (\mu - L^2/\nu^2)u^2 + (\nu^2 v - Lu)^2/\nu^2$$

(for $\nu \neq 0$), from the last estimate we obtain

$$V'(t,x_t,y_t) \leq -\left[h(2b(t)/h - \nu^2(t)) - L^2\int_{-h}^{0}(1/\nu^2(t+u))du\right]y^2(t). \tag{C9}$$

PROPOSITION C1. If

(i) the function $b(t) - Lh$ is positive in measure then the zero solution of (C1) is stable, and for every solution $x(t)$ with sufficiently small initial function it follows that $\lim_{t\to\infty} x'(t) = 0$ and $\lim_{t\to\infty} x(t)$ exists and is finite.

If, in addition, for some $K > 0$ and M the inequality

$$\text{(ii)} \quad \int_{t-h}^{t} \sup\{\phi(y,s)/y : 0 < |y| \le K\} ds \le M$$

holds, then the zero solution of (C1) is asymptotically stable.

PROOF. In (C8) take $\nu^2(t) \equiv L$ so that (C9) yields

$$V'(t, x_t, y_t) \le -2(b(t) - Lh)y^2(t) \le 0.$$

Theorem 1(b) then shows that the zero solution is stable. Let $0 < K < H$, $t_0 \le R_+$, and take $\delta = \delta(K, t_0)$ to be the positive number from the definition of stability.

For any arbitrary solution $(x(t), y(t))$ with $\|(x_{t_0}, y_{t_0})\| < \delta$ define the functional $D(t, x_t, y_t) = y^2(t)$, which is bounded on $[t_0, \infty)$. By Theorem 4 we have $\lim_{t\to\infty} \int_{t-h}^{t} y^2(s)ds = 0$. Next, we show that $\lim_{t\to\infty} y(t) = 0$.

If this is not true, then there are $\epsilon > 0$ and $t_i' < t_i'' < t_{i+1}'$ $(i = 1, 2, \dots)$ such that $t_i' \to \infty$ $(i \to \infty)$, $|y(t_i')| = \epsilon$, $|y(t_i'')| = 2\epsilon$, and $\epsilon \le |y(t)| \le 2\epsilon$ on $[t_i', t_i'']$ for all i. Since $\int_{t-h}^{t} y^2(s)ds \to 0$ as $t \to \infty$, it follows that $t_i'' - t_i' \to 0$ as $i \to \infty$. Since $y(t)$ is bounded and $x'(t) = y(t)$ we have $x(t_i'') - x(t_i') = \int_{t_i'}^{t_i''} y(t)dt \to 0$ as $i \to \infty$. Since $x(t)$ is bounded, it follows that

$$F(x(t_i'')) - F(x(t_i')) = f(x(\xi_i))(x(t_i'') - x(t_i')) \to 0$$

as $i \to \infty$. (Here, $x(\xi_i)$ is between $x(t_i'')$ and $x(t_i')$.) On the other hand,

$$0 = \lim_{t\to\infty} \left[V(x_{t_i''}, y_{t_i''}) - V(x_{t_i'}, y_{t_i'})\right] = 2 \lim_{i\to\infty} \left[F(x(t_i'')) - F(x(t_i'))\right] + 3\epsilon^2 = 3\epsilon^2, \qquad \text{(C10)}$$

a contradiction.

Thus we have proved that $y(t) \to 0$ as $t \to \infty$ which, combined with $V(x_t, y_t) \to V_0$ and

$$\int_{-h}^{0} \int_{t-u}^{t} \nu^2(s)y^2(s)ds\, du \le hL^2 \int_{t-h}^{t} y^2(s)ds \to 0$$

implies the existence of $\lim_{t\to\infty} F(x(t))$. But F is strictly increasing so $\lim_{t\to\infty} x(t) = x_0$ exists.

In order to prove the last assertion of the proposition, suppose that (ii) holds and $x_0 \neq 0$. Then integrating the second equation of (C2) and using the condition (ii) we obtain

$$|y(t+h) - y(t)| \geq f(x_0)(h/2) - (M + Lh)\|y_t\|$$

for sufficiently large t. But $y(t) \to 0$ as $t \to \infty$ and $|f(x_0)| > 0$, so the last estimate yields a contradiction which completes the proof.

We may eliminate the condition $b(t) \geq Lh$ by allowing ν^2 in (C8) to vary as a function of s. We also note that Yoshizawa [19] has allowed ν^2 to vary.

PROPOSITION C2. Suppose that there exists a measurable function $\beta : R_+ \to R_+$ satisfying

(i) $0 < \beta(t) \leq b(t)$, $\int_{t-h}^{t} \beta^2(s)ds$ is bounded on R_+ and

(ii) the function $\beta(t) - L^2 h \int_{t-h}^{t} [1/\beta(s)]ds$ is PIM.

Then the zero solution of (C1) is stable and for every solution with sufficiently small initial functions, it follows that $\lim_{t\to\infty} x'(t) = 0$ and $\lim_{t\to\infty} x(t)$ exists and is finite. If, in addition, condition (ii) of Proposition C1 is satisfied, then the zero solution is asymptotically stable.

PROOF. In (C8) define $\nu^2(t) = \beta(t)/h$. Then from (C9) we have

$$V'(t, x_t, y_t) \leq -\left[\beta(t) - L^2 h \int_{-h}^{0} [1/\beta(t+u)]du\right] y^2(t) \leq 0.$$

In order to be able to repeat the proof of Proposition C1 we need only show that for any solution of (C2) we have

$$\lim_{t\to\infty} V(t, x_t, y_t) = \lim_{t\to\infty} [2F(x(t)) + y^2(t)] \tag{C11}$$

whenever $\int_{t-h}^{t} y^2(s)ds \to 0$ and $|y(t)| \leq K$ on R_+ (recall that (C11) was used in (C10)). But under these conditions we have

$$\left[(1/h)\int_{-h}^{0}\int_{t+u}^{t}\beta(s)y^2(s)ds\,du\right]^2 \le K^2\left[\int_{t-h}^{t}\beta(s)|y(s)|ds\right]^2$$
$$\le K^2\int_{t-h}^{t}\beta^2(s)ds\int_{t-h}^{t}y^2(s)ds \to 0 \quad \text{as} \quad t\to\infty$$

because of (i). Thus, we have (C11) and the proof is complete.

The following corollary shows that Proposition C2 allows $b(t)\equiv Lh$; in fact, it allows $b(s) < Lh$ on intervals $[t, t+\xi]$ (for $0<\xi=\text{constant}$) for arbitrarily large t. In this corollary we represent β in the form $\beta(t) = Lh+\gamma(t)$.

COROLLARY. Let the measurable function $\gamma : R_+ \to R$ and let $\alpha_1 \in (0,1)$, $\alpha_2 > 0$ be given such that

$$\gamma(t) + (1/h)\int_{-h}^{0}\gamma(t+u)du \ge \alpha_2 Lh$$

and either

$$0 \le \gamma(t) \le \alpha_1 Lh \quad \text{for} \quad t\in R_+ \quad \text{and} \quad \alpha_2 > \alpha_1^2$$

or

$$|\gamma(t)| \le \alpha_1 Lh \quad \text{for} \quad t\in \mathbf{R}_+ \quad \text{and} \quad \alpha_2 > \alpha_1^2/(1-\alpha_1)$$

hold. Suppose also that the function ϕ in (C1) satisfies

$$\phi(y,t)/y \ge Lh + \gamma(t) \quad \text{for} \quad y\ne 0 \quad \text{and} \quad t\in R_+$$

and

$$\int_{t-h}^{t}\sup\{\phi(y,s)/y : 0<|y|\le K\}ds \le M$$

for all $t\in R_+$ and some $K>0$ and M. Then the zero solution of (C1) is asymptotically stable.

PROOF. Using the identity $1/(1+x) = 1 - x + x^2/(1+x)$ for $|x| \le \alpha_1$ we can write

$$Lh + \gamma(t) - L^2 h \int_{-h}^{0} \big[1/(Lh + \gamma(t+u))\big] du = \gamma(t) + (1/h) \int_{-h}^{0} \gamma(t+u) du + R(t)$$

where

$$|R(t)| \le \begin{cases} Lh\alpha_1^2 & \text{if } \gamma(t) \ge 0 \\ Lh\alpha_1^2/(1-\alpha_1) & \text{otherwise.} \end{cases}$$

Thus, under the conditions of the corollary of the conditions of Proposition C2 are met, so the proof is complete.

For example, the choices

$$\gamma(t) = \begin{cases} Lh/2 & \text{for } 0 \le t \le 2h/3 \\ 0 & \text{for } 2h/3 < t < h \\ \gamma & \text{is } h\text{-periodic,} \end{cases}$$

$\alpha_1 = 1/2$, $\alpha_2 = 1/3$ (so that $\alpha_2 > \alpha_1^2$), or

$$\gamma(t) = \begin{cases} Lh/16 & \text{for } 0 \le t \le 3h/4 \\ -Lh/32 & \text{for } 3h/4 < t < h \\ \gamma & \text{is } h\text{-periodic,} \end{cases}$$

$\alpha_1 = 1/16$, $\alpha_2 = 1/32 \cdot 4$ (so that $\alpha_2 > \alpha_1^2/(1-\alpha_1)$) are suitable for the corollary.

REMARK. Consider the sunflower equation (C7) with varying damping coefficient:

$$x''(t) + b(t)x'(t) + L \sin x(t-h) = 0. \tag{C7$'$}$$

Applying Proposition C1 to this special case of (C1) (keeping x small so that $x \sin x > 0$ for $x \ne 0$) we obtain the following assertion: If $b(t) - Lh$ is PIM and $\int_{t-h}^{t} b(s)ds$ is bounded on R_+, then the zero solution of (C7$'$) is asymptotically stable.

This generalizes some statements of [4; pp. 151–153] and the first assertion of Somolinos' theorem (see (i) in Example C in Section 2) to the nonautonomous case. But comparing our corollary with this assertion one can observe that the following interesting problem remains open: Is the zero solution of (C7$'$) asymptotically stable if $b(t) - b_0$ is PIM and $\int_{t-h}^{t} b(s)ds$

is bounded on $\mathbf{R}_2$, where $b_0 = Lh(\sin\xi)/\xi < Lh$ and ξ is the root of $\sigma^2/Lh^2 = \cos\sigma$ in $(0, \pi/2)$?

EXAMPLE B revisited. Consider now the nonautonomous equation

$$x'(t) = -\alpha(t) \int_{t-h}^{t} a(t-s)g(x(s))ds, \tag{B11}$$

where the functions a and g satisfy the conditions with (B1), while $\alpha : R_+ \to R_+$ is differentiable and satisfies

$$\alpha'(t)(-a'(r)) - \alpha(t)\alpha''(r) \le 0 \quad \text{for} \quad t \in \mathbf{R}_+, \quad r \in [0, h]$$

and

$$\lim_{t\to\infty} \alpha(t) = \alpha_0 \quad \text{exists.}$$

THEOREM B2. (1) If $\alpha_0 = 0$, then for every solution $x(t)$ of (B11) the limit $\lim\limits_{t\to\infty} x(t)$ exists and is finite.

(2) Suppose that $\alpha_0 > 0$.

(a) If there is a $\lambda \in [0, h]$ with $a''(\lambda) \neq 0$ and

$$\lim_{t\to\infty} \int_{t-h}^{t} |\alpha'(s)|ds = 0. \tag{B12}$$

then every solution of (B11) tends to one of the zeros of g as $t \to \infty$.

(b) If $a''(\lambda) \equiv 0$ on $[0, h]$, then for every solution $x(t)$ of (B11) there is an h-periodic solution $z(t)$ of the equation

$$z''(t) + \alpha_0 a(0) g(z(t)) = 0 \tag{B13}$$

such that the functions $(x(t), x'(t))$ and $(z(t), z'(t))$ have the same positive limit sets.

PROOF. Consider the Liapunov functional

$$V(t, \phi) = G(\phi(0)) - [\alpha(t)/2] \int_{-h}^{0} a'(-s) \left[\int_{s}^{0} g(\phi(u))du \right]^2 ds.$$

A computation yields

$$V'(t,x_t) = [\alpha(t)/2]a'(h)\Big[\int_{t-h}^{t} g(x(u))du\Big]^2 - (1/2)\int_{t-h}^{t} A(t,s)\Big[\int_{s}^{t} g(x(u))du\Big]^2 ds$$

where $A(t,s) := \alpha'(t)a'(t-s) + \alpha(t)a''(t-s) \geq 0$. Since $G(x) \to \infty$ as $|x| \to \infty$ and $V'(t,x_t) \leq 0$, every solution is bounded on $[t_0, \infty)$.

For an arbitrary solution x, define

$$D_1(t,x_t) = \int_{t-h}^{t} A(t,s)\Big[\int_{s}^{t} g(x(u))du\Big]^2 ds$$

and

$$D_2(t,x_t) = \int_{t-h}^{t} g(x(u))du\Big]^2.$$

If $\alpha_0 > 0$, then $\alpha(t)$ is PIM so by Theorem 4 we have

$$\lim_{t\to\infty} \int_{t-h}^{t} D_1(s,x_s)ds = 0 \tag{B14}$$

and

$$\lim_{t\to\infty} \int_{t-h}^{t} D_2(s,x_s)ds = 0, \quad \text{for} \quad \alpha_0 a'(h) \neq 0. \tag{B15}$$

By differentiation and integration by parts it can be shown that the solution $x(t)$ satisfies

$$x''(t) + a(0)\alpha(t)g(x(t)) = f(t) \tag{B16}$$

where

$$f(t) := -a'(h)\alpha(t)\int_{t-h}^{t} g(x(u))du + \int_{t-h}^{t} A(t,s)\int_{s}^{t} g(x(u))du\,ds.$$

Using the Schwarz inequality, (B14) and (B15) yield

$$\lim_{t\to\infty} \int_{-K}^{K} |f(t+T)|dt = 0 \tag{B17}$$

for arbitrary K; thus, (B13) is the single limiting equation of (B16) (see [2]).

Let p be an arbitrary point of the positive limit set $\Omega(x) \subset R$ of the solution $x(t)$, and let $\{t_n\}$ be a sequence with $t_n \to \infty$, $x(t_n) \to p$ (as $n \to \infty$). Since $x(t)$ and $x'(t)$ are bounded on R_+ and since (B16) and (B17) imply that

$$\lim_{t-s\to 0} |x'(t) - x'(s)| = \lim_{t-s\to 0} \left| \int_s^t x''(u)du \right| = 0,$$

it follows that $\{x(t_n + s)\}$ and $\{x'(t_n + s)\}$ are uniformly bounded and equicontinuous for $|s| \le K$. By the Arzela-Ascoli lemma, it can be assumed that $x(t_n + s) \to \psi(s)$ and $x'(t_n + s) \to \psi'(s)$ as $n \to \infty$ uniformly for $s \in [-K, K]$. As is known ([2], Theorem 7.3), ψ is the solution of the initial value problem

$$x''(t) + \alpha_0 a(0) g(x(t)) = 0, \quad x(0) = p, \quad x'(0) = \psi'(0). \tag{B18}$$

This means that $\Omega(x, x')$ consists of complete trajectories of solutions of (B13).

(1) If $\alpha_0 = 0$, then $\psi(t) = p + c_1 t$. But $\Omega(x, x') \subset R^2$ is invariant with respect to (B13) ([2], Theorem 7.3) so $(p + c_1 t, c_1) \in \Omega(x, x')$ for all $t \in [-K, K]$. As K is arbitrary and $x(t)$ is bounded, we get $c_1 = \psi'(0) = 0$. In other words, the positive limit set $\Omega(x, x')$ is a compact connected subset of the x-axis $\{(x, y) : y = 0\}$. We now show that Ω consists of the single point $(p, 0)$.

Suppose that $q \in \Omega(x)$ so that there is a sequence $\{s_n\}$ with $s_n \to \infty$ and $x(s_n) \to q$ as $n \to \infty$. Since $\lim_{t\to\infty} V(t, x_t) = \lim_{t\to\infty} G(x(t))$ exists, $G(q) = G(p)$. Hence, G is constant on the connected set Ω. As the zeros of g are isolated, this means that $\Omega(x) = \{p\}$.

(2) (a) By (B14) we have

$$\lim_{n\to\infty} \int_{-h}^0 \int_{-h}^0 [\alpha'(t_n + u) a'(-s) + \alpha(t_n + u) a''(-s)] \left[\int_s^0 g(x(t_n + u + v)) dv \right]^2 ds\, du = 0. \tag{B19}$$

Since $x(t)$ is bounded

$$\left| \int_{-h}^0 \alpha'(t_n + u) \left[\int_s^0 |g(x(t_n + u + v)) dv \right]^2 du \right| \le \int_{-h}^0 |\alpha'(t_n + u)| \left[\int_{-h}^0 |g(x(t_n + u + v))| dv \right]^2 du$$

$$\le c_1 \int_{t_n - h}^{t_n} |\alpha'(u)| du \quad \text{where} \quad c_1 \quad \text{is constant.}$$

Now the last term tends to zero uniformly for $s \in [-h, 0]$ as $n \to \infty$. Hence, from (B19) we obtain

$$\int_{-K}^{K} \int_{-h}^{0} a''(-s) \left[\int_{s}^{0} g(\psi(u+v))dv \right]^2 ds\, du = 0.$$

Therefore, there exists a pair $\xi_1 < \xi_2$ with $-h < \xi_1 < \xi_2 < 0$ such that $\{s \in (\xi_1, \xi_2),$ $u \in [-K, K]\}$ imply that

$$\int_{s}^{0} g(\psi(u+v))dv = \int_{u+s}^{u} g(\psi(v))dv = 0.$$

This means that $g(\psi(s)) \equiv 0$ for all s. Since the zeros of g are isolated we get $\psi(s) \equiv p$ and $g(p) = 0$.

(2) (b) Suppose now that $a''(u) = 0$ for all $u \in [0, h]$. We have already proved that $\Omega(x, x')$ consists of complete trajectories of (B13). We have to show that $\Omega(x, x')$ may contain only the trajectory of a single h-periodic solution of (B13).

We know that $a'(h) \neq 0$, since otherwise $a(u) \equiv 0$. From (B15) we obtain

$$\lim_{n\to\infty} \int_{t_n-h}^{t_n} \left[\int_{s-h}^{s} g(x(u))du \right]^2 ds = \int_{-h}^{0} \left[\int_{s-h}^{s} g(\psi(u))du \right]^2 ds = 0$$

and hence $\int\limits_{s-h}^{s} g(\psi(u))du = 0$ provided that ψ is defined on $[s-h, s]$. By equation (B13) we have $\psi'(s) - \psi'(s-h) = 0$, so $\psi(s)$ is h-periodic because $\psi(s)$ is bounded.

As ψ is a solution of equation (B13) it satisfies the identity

$$[\psi'(s)]^2/2 + \alpha_0 a(0) G(\psi(s)) \equiv \alpha_0 a(0) G(\psi(0)). \qquad \text{(B20)}$$

The limit set $\Omega(x)$ can also be located by the Liapunov functional V since

$$\lim_{t\to\infty} V(t, x_t) = \lim_{t\to\infty} \left\{ G(x(t)) - [\alpha(t)/2] \int_{-h}^{0} a'(-s) \left[\int_{s}^{0} g(x(t+u))du \right]^2 ds \right\}$$
$$= G(\psi(0)) - [\alpha_0/2] a'(h) \int_{-h}^{0} \left[\int_{s}^{0} g(\psi(u))du \right]^2 ds =: V_*(\psi).$$

That is, the functional $V_* : C_H \to R_+$ takes the same value at all the h-periodic solutions of (B13) whose trajectories lie in $\Omega(x, x')$.

Using these facts we can complete the proof in the same way as it is done in Hale's autonomous case.

We now augment the conditions of Theorem 4 so that it will guarantee U.A.S.

THEOREM 5. Suppose that $D, V : R_+ \times C_H \to R_+$ are continuous, $\eta : R_+ \to R_+$ is PIM, and the following conditions are satisfied.

(i) $W_1(|x(t)|) \leq V(t, x_t) \leq W_2(|x(t)|) + W_3\big[\int\limits_{t-h}^{t} D(s, x_s)ds\big]$;

(ii) $V'_{(1)}(t, x_t) \leq -\eta(t)W_4(D(t, x_t))$;

(iii) $D(t, \phi) \leq W_5(\|\phi\|)$;

(iv) for some $K \in (0, H)$ there is a wedge W_K such that $[t \in R_+, u : [-2h, 0] \to R^n$ is continuous, $|u(s)| \leqq K$ for $s \in [-2h, 0]$ imply

$$W_K\big(\inf\{|u(r)| : -h \leqq r \leqq 0\}\big) \leqq \int_{-h}^{0} D(t + s, u_s)ds.$$

Then $x = 0$ is U.A.S.

PROOF. There is a W_6 with $V(t, \phi) \leq W_6(\|\phi\|)$ so $x = 0$ is U.S. by Theorem 1(b). Let δ be that of U.S. for K and let $\gamma > 0$ be given. We must find T such that $[t_0 \in R_+,\ \|\phi\| < \delta,$ $t \geq t_0 + T]$ imply that $|x(t, t_0, \phi)| < \gamma$.

Find $\xi = \xi(\gamma) > 0$ with $W_2(\xi) + W_3(\xi) < W_1(\gamma)$. Consider the intervals

$$I_j = [t_0 + jh,\ t_0 + (j + 1)h], \quad j = 0, 1, 2, \dots\ .$$

Suppose that for each j there is a $t_j \in I_j$ with $\int_{t_j-h}^{t_j} D(s, x_s)ds \geq \min\{\xi, W(\xi)\}$. By condition (iii) we have $0 \leq D(t, x_t) \leq W_5(K)$ for $t \in R_+$. From Lemma 2 it follows that there exists $T_*(\xi)$ and $\beta(\xi)$ with

$$\int_{t_j-h}^{t_j} \eta(s)W_4(D(s, x_s))ds \geq \beta(\xi) \quad \text{if} \quad t_j \geq T_*(\xi).$$

Let $\mathcal{F} = \mathcal{F}(\xi)$ be a natural number with $2\mathcal{F}h > T_*(\xi)$. Then

$$[N - \mathcal{F}(\xi)]/\beta(\xi) \le \sum_{j=\mathcal{F}(\xi)}^{N} \int_{t_{2j}-h}^{t_{2j}} \eta(s) W_4(D(s, x_s)) ds \le V(t_0, \phi) \le W_6(\delta).$$

Hence, there exists $N = N(\xi)$ such that t_j fails to exist in some I_j with $j \le 2N$. Thus $\int\limits_{t-h}^{t} D(s, x_s) ds < \min\{\xi, W_K(\xi)\}$ for all $t \in I_j$; of course, I_j depends on the solution $x(t, t_0, \phi)$, but for any solution $j \le 2N$, and N depends only on $\xi = \xi(\gamma)$. However, by definition of W_K there is a $t^* \in I_j$ with $|x(t^*)| \le \xi$. Hence, for $t \ge t^*$ we have

$$W_1(|x(t)|) \le V(t, x_t) \le V(t^*, x_{t^*}) \le W_2(\xi) + W_3(\xi) \le W_1(\gamma),$$

and for $T(\gamma) = 2N(\xi) = 2N(\xi(\gamma))$ the proof is complete.

Setting $D(s, \phi) = |\phi(0)|^2$ we obtain a conceptually simpler result.

COROLLARY. Let η be PIM and $V : R_+ \times C_H \to R_+$ be continuous with

(i) $W_1(|\phi(0)|) \le V(t, \phi) \le W_2(|\phi(0)|) + W_3(|||\phi|||)$ and

(ii) $V'_{(1)}(t, x_t) \le -\eta(t) W_4(|x(t)|)$.

Then $x = 0$ is U.A.S.

EXAMPLE 1 revisited. Consider again the equation

$$x'(t) = -a(t)x(t) + b(t) \int_{t-h}^{t} x(u) du$$

in the case when $a(t)$ may equal $\int_t^{t+h} |b(s)| ds$ for arbitrarily large values of t. In this case the method of Section 3 does not work. However, using Theorems 4 and 5 we can prove the following assertions.

A. If

(i) $a(t) - \int\limits_{t}^{t+h} |b(s)| ds$ is PIM and

(ii) $\int\limits_{t}^{t+h} |b(s)| ds$ is bounded on R_+

then $x = 0$ is U.A.S.

B. If

(i) $0 < \int_t^{t+h} |b(s)|ds$ is bounded on R_+ and

(ii) $\left[a(t)/ \int_t^{t+h} |b(s)|ds\right] - 1$ is PIM,

then every solution has a finite limit as $t \to \infty$. If, in addition,

(iii) $\limsup_{t\to\infty} \int_{t-h}^{t} \int_{s}^{s+h} |b(u)|du\, ds > 0$,

then $x = 0$ is A.S. If (i)–(ii) are satisfied and, in addition,

(iii′) $\liminf_{t\to\infty} \int_{t-h}^{t} \int_{s}^{s+h} |b(u)|du\, ds > 0$,

then $x = 0$ is U.A.S.

PROOF. Define

$$
\begin{aligned}
V(t, x_t) &= |x(t)| + \int_{-h}^{0} \int_{t+s}^{t} |b(u-s)|\,|x(u)|du\, ds = |x(t)| + \int_{t-h}^{t} \left[\int_{t}^{u+h} |b(s)|ds\right]|x(u)|du \\
&\le |x(t)| + \int_{t-h}^{t} \left[\int_{u}^{u+h} |b(s)|ds\right]|x(u)|du \le |x(t)| + K\int_{t-h}^{t} |x(u)|du
\end{aligned}
$$

where $\int_t^{t+h} |b(s)|ds \le K$ for $t \in R_+$, and K is constant. We then have

$$
V'(t, x_t) \le -\left[a(t) - \int_t^{t+h} |b(s)|ds\right]|x(t)|.
$$

Assertion A follows from Theorem 5 with $D(t, x_t) = |x(t)|$. In order to prove B set $D(t, x_t) = \int_t^{t+h} |b(s)|ds|x(t)|$. By Theorem 4, for every solution $x(t)$ we obtain

$$
\lim_{t\to\infty} \int_{t-h}^{t} \left[\int_{s}^{s+h} |b(u)|du\right]|x(s)|ds = 0.
$$

On the other hand, $\lim_{t\to\infty} V(t, x_t) = v_0$ exists and

$$
\lim_{t\to\infty} \int_{t-h}^{t} \left[\int_{t}^{u+h} |b(s)|ds\right]|x(u)|du = 0;
$$

hence, $\lim_{t\to\infty} |x(t)| = v_0$ exists also.

If condition (iii) is satisfied, then $\lim_{t\to\infty} x(t) = 0$. The final assertion follows from Theorem 5.

The corollary after Theorem 5 tells us that if we are able to use the L^2-norm in the upper bound on V, then we can conclude U.A.S. without the classical requirement of $F(t, \phi)$ bounded for ϕ bounded. This can be done because when we integrate $V'_{(1)}(t, x_t) \leq -\eta(t)W_4(|x(t)|)$ we can, in effect, pass the integral inside W_4 obtaining $W_4\big[\int_{t-h}^{t} |x(s)|ds\big]$ which we can then compare with the upper bound of $W_3(|||x_t|||)$ on V. The next result tells us that if we are forced to use the supremum norm in our upper bound on V then it is satisfactory to have $V'_{(1)}(t, x_t) \leq -W_4(|x'|)$ because we can integrate the last expression, pass the integral inside W_4, and obtain an expression which can be compared with the supremum norm upper bound on V. We have noticed in examples that investigators have frequently had an L^2-upper bound on V and have had $V' \leq -W(|x'|)$, but have made no effort to use these facts and have, consequently, been forced to ask $F(t, \phi)$ bounded for ϕ bounded.

THEOREM 6. Suppose that $V : R_+ \times C_H \to R_+$ is continuous with

(i) $W_1(|\phi(0)|) \leq V(t, \phi) \leq W_2(\|\phi\|)$ and

(ii) $V'_{(1)}(t, x_t) \leq -\eta_1 W_3(|x'(t)|) - \eta_2(t)W_4(|x(t)|)$ where $\eta_1 > 0$ is constant, $\lim_{S\to\infty} \int_{t_*}^{t_*+S} \eta_2(s)ds = \infty$ uniformly with respect to t_*, and there are $\alpha > 0$, r_0 such that $r > r_0$ implies $W_3(r) \geq \alpha r$.

Then $x = 0$ is U.A.S.

PROOF. By Theorem 1(b) the zero solution is U.S. Let $0 < H' < H$ and take $\delta = \delta(H')$ of U.S. Let $\gamma > 0$ be given and find $\xi > 0$ with $W_2(\xi) < W_1(\gamma)$. We must find $T = T(\gamma) > 0$ such that $[t_0 \in R_+,\ \|\phi\| < \delta,\ t \geq t_0 + T]$ imply that $|x(t, t_0, \phi)| < \gamma$.

Let $t_0 \in R_+$ be arbitrary, $\|\phi\| < \delta$, $x(t) = x(t, t_0, \phi)$, and $v(t) = V((t, x_t))$. First we prove the existence of an $L = L(\gamma) > 0$ such that for each $t_1 \geq t_0$ the interval $[t_1, t_1 + L]$ contains a point t with $|x(t)| < \xi/2$.

By assumption on η_2 there is an $L = L(\gamma)$ such that $\int\limits_{t_1}^{t_1+L} \eta_2(s)ds > W_2(\delta)/W_4(\xi/2)$ for all $t_1 \in R_+$. If $|x(t)| \geq \xi/2$ were true for all $t \in [t_1, t_1 + L]$, then we would have

$$0 \leq v(t_1 + L) \leq v(t_0) - \int_{t_1}^{t_1+L} \eta_2(s)W_4(|x(s)|)ds \leq W_2(\delta) - W_4(\xi/2) \int_{t_1}^{t_1+L} \eta_2(s)ds < 0,$$

a contradiction.

Consider the intervals $I_j = [t_0 + jL,\ t_0 + (j + 1)L]$ and find $t_j \in I_j$ with $|x(t_j)| < \xi/2$. Suppose that for each such t_j we also have $\|x_{t_j}\| > \xi$. Then there is an $h_j \in [0, h]$ with $|x(t_j - h_j)| \geq \xi$ and so $\xi/2 \leq \big|\int\limits_{t_j-h_j}^{t_j} x'(s)ds\big| \leq \int\limits_{t_j-h}^{t_j} |x'(s)|ds$. Define

$$p_1(t) = \begin{cases} |x'(t)| & \text{if } |x'(t)| \leq r_0 \\ 0 & \text{otherwise} \end{cases}$$

and $p_2(t) = |x'(t)| - p_1(t)$. If $\int\limits_{t_j-h}^{t_j} p_1(t)dt \geq \xi/4$ then by Lemma 2 there are $\beta_1 = \beta_1(\gamma)$ and $N_1 = N_1(\gamma)$ with $\int\limits_{t_j-h}^{t_j} \eta_1 W_3(p_1(s))ds \geq \beta_1$ for $j \geq N_1$. If $\int\limits_{t_j-h}^{t_j} p_2(t)dt \geq \xi/4$ then, obviously, $\int\limits_{t_j-h}^{t_j} \eta_1 W_3(p_2(s))ds \geq \eta_1\alpha\xi/4 =: \beta_2 > 0$. In any case we have

$$\int_{t_j-h}^{t_j} \eta_1 W_3(|x'(t)|)dt \geq \min\{\beta_1, \beta_2\} =: \beta = \beta(\gamma) > 0$$

for $j \geq N_1$. This means that v decreases at least β units on each I_j. Since $v(t) \leq v(t_0) \leq W_2(\delta)$, there is an $N = N(\gamma) > N_1$ such that the existence of such a t_j fails in some I_j with $j \leq N$. Hence, there is a t_j with $j \leq N$ for which $\|x_{t_j}\| < \xi$. This means that for $t \geq t_0 + NL$ we have $W_1(|x(t)|) \leq v(t) \leq v(t_j) \leq W_2(\xi) < W_1(\gamma)$, and the proof is complete.

EXAMPLE D re-revisited. Consider once more the equation (D1) with

$$V(t, x_t) = |x(t)| + M \int_{t-h}^{t} |b(u + h)|\,|x(u)|du.$$

Then

$$V'(t, x_t) \leq \big[- a(t) + M|b(t + h)|\big]|x(t)| + (1 - M)|b(t)|\,|x(t - h)|.$$

Since $|b(t)|\,|x(t-h)| \geq |x'(t)| - a(t)|x(t)|$, we have

$$V'(t, x_t) \leq -(M-1)|x'(t)| + \big[(M-2)a(t) + M|b(t+h)|\big]|x(t)|.$$

Let $M = 2\beta/(\beta+1)$ for $\beta > 1$. Then

$$V'(t, x_t) \leq -\eta_1(t)|x'(t)| - \eta_2(t)|x(t)|$$

where $\eta_1 = (\beta-1)/(\beta+1) = \text{const.} > 0$ and

$$\eta_2(t) = [2/(\beta+1)](a(t) - \beta|b(t+h)|).$$

Hence, if we ask that for some $\beta > 1$ the relations

$$\eta(t) = a(t) - \beta|b(t+h)| \geq 0, \quad \lim_{s\to\infty} \int_{t_*}^{t_*+S} \eta(t)dt = \infty \tag{D6}$$

hold uniformly with respect to $t_* \in R_+$, then (ii) in Theorem 6 is satisfied. Consequently, if $\int_t^{t+h} |b(s)|ds$ is bounded on R_+ and (D6) is satisfied, then $x = 0$ is U.A.S.

The next theorem will be useful in cases when the Liapunov functional is the sum of a function and a functional in which the latter does not increase too fast along solutions.

THEOREM 7. Let $Z, V : R_+ \times C_H \to R_+$ with Z locally Lipschitzian and V continuous such that the following are satisfied:

(i) $W_1(|\phi(0)|) + Z(t,\phi) \leq V(t,\phi) \leq W_2(|\phi(0)|) + Z(t,\phi) \leq W_3(\|\phi\|)$;

(ii) $V'_{(1)}(t, x_t) \leq -\eta(t)W_4(|x(t)|)$
where $\eta : R_+ \to R_+$ is UWIP(δ, h) for every $\delta > 0$;

(iii) for some $H' \in (0, H)$ the function

$$\Gamma_{H'}(t) = \int_0^t \sup\{Z'_{(1)}(s,\phi) : \phi \in C_H,\ \|\phi\| \leq H'\}ds$$

is uniformly continuous in R_+.

Then $x = 0$ is U.A.S.

PROOF. Since $V(t,\phi) \leq W_3(\|\phi\|)$ and $V'(t, x_t) \leq 0$, it follows from Theorem 1(b) that $x = 0$ is U.S. Let δ be that of U.S. for H' and let $\gamma > 0$ be given. We must find T such that $[t_0 \in R_+,\ \phi \in C_\delta,\ t \geq t_0 + T]$ imply that $|x(t, t_0, \phi)| < \gamma$. For this $\gamma > 0$ find ξ of U.S. Consider the intervals $I_j = [t_0 + jh,\ t_0 + (j+1)h]$ and suppose that in each I_j there is a t_j with $|x(t_j)| \geq \xi$. Find $\alpha = \alpha(\gamma) > 0$ such that $W_2(\alpha) < W_1(\xi)/2$. Because $0 \leq V(t, \phi) \leq W_3(\|\phi\|)$ and $V'_{(1)}(t, x_t) \leq -\eta(t)W_4(|x(t)|)$ with η being UWIP there is a $T_1 > 0$ such that $|x(t)| \geq \alpha$ fails at some point in every interval of length T_1. Find a sequence $\{s_i\}$ such that $t_i < s_i$, $x(s_i)| = \alpha$ and $|x(t)| \geq \alpha$ for $t_i \leq t \leq s_i$. How large is $s_i - t_i$? We have

$$W_1(|x(t)|) \leq V(t, x_t) - Z(t, x_t) \leq W_2(|x(t)|),$$

$$W_1(\xi) \leq V(t_i, x_{t_i}) - Z(t_i, x_{t_i}),$$

and

$$V(s_i, x_{s_i}) - Z(s_i, x_{s_i}) \leq W_2(\alpha) < W_1(\xi)/2.$$

Thus, on each interval $[t_i, s_i]$ the function $V(t, x_t) - Z(t, x_t)$ decreases by at least $W_1(\xi)/2 =:$ β. If on $[t_i, s_i]$, $V(t, x_t)$ has not decreased by $\beta/2$, then $Z(t, x_t)$ has increased by at least $\beta/2$. But $\beta/2 \leq Z(s_i, x_{s_i}) - Z(t_i, x_{t_i}) \leq \Gamma_{H'}(s_i - t_i)$. By condition (iii) there is a $\beta_1 = \beta_1(H') > 0$ such that $s_i - t_i \geq \beta_1$.

If $i_1 < i_2 < \cdots < i_k < \cdots$ is a suitable sequence of positive integers then

$$s_i - t_{i_k} \geq \beta_1, \qquad t_{i_k} < s_{i_k} < t_{i_{k+1}} < s_{i_k} + h.$$

Denote the subsequences $\{t_{i_k}\}$ and $\{s_{i_k}\}$ by $\{t_i\}$ and s_i respectively. Then

$$W_3(H') \geq V(t_N, x_{t_N}) - V(t_0, x_{t_0}) \geq W_4(\alpha) \sum_{i=1}^{N} \int_{t_i}^{s_i} \eta(t)dt.$$

The function η is UWIP(β_1, h) and, hence, there is a $Q = Q(\gamma)$ such that for all $P > Q$

$$\int_{[t_1, t_1+P] \cap I} \eta(t)dt > W_3(H')/W_4(\alpha)$$

where $I = \bigcup_{i=1}^{\infty} [t_i, s_i]$. Choose $T = T(\gamma) = Q(\gamma) + h$ to complete the proof.

An application can be found in the next section.

5 Applications to the Equation $x'(t) = -a(t)f(x(t)) + b(t)\int\limits_{t-h}^{t} \lambda(s)g(x(s))ds$

In this section we give several sufficient conditions for the asymptotic stability and uniform asymptotic stability of the zero solution of the nonlinear scalar equation in the title as consequences of our theorems in Sections 3 and 4. The results will show that these theorems are independent and complementary to one another. Moreover, it will be illustrated how to get complementary sufficient conditions for the same property of the solutions of the same equation by use of different Liapunov functionals.

We assume that the functions $a, b, \lambda : R_+ \to R$, $f, g : R \to R$ are continuous, $xf(x) \geq 0$, and $|g(x)| \leq c|f(x)|$ for some constant $c \geq 0$.

In order to compare the results we list the hypotheses to be used:

(H1) $f(x) \neq 0$ for $x \neq 0$;

(H2) $g(x) \neq 0$ for $x \neq 0$;

(H3) for some $\alpha \geq 1$

$$\gamma_\alpha(t) := a(t) - \alpha c|\lambda(t)| \int_t^{t+h} |b(u)|du \geq 0 \quad \text{for} \quad t \geq 0;$$

(H4) for some $\alpha \geq 1$, $\int\limits_0^\infty \gamma_\alpha(t)dt = \infty$;

(H5) for every $\delta > 0$ the function $\gamma_1(t)$ is UWIP(δ, h);

(H6) $\lambda(t) \neq 0$ for $t \geq 0$ and $\gamma_1(t)/|\lambda(t)|$ is PIM;

(H7) $\gamma_1(t)$ is PIM;

(H8) for some $h_1 \in (0, h)$ the function $b(t)$ is UWIP$(h_1, 4h)$;

(H9) for some $h_1 \in (0, h)$ the function $b(t)/B(t+h)$ is IP(h_1), where

$$B(t) = \sup\left\{ \int_s^{s+h} |b(u)|du : 0 \leq s \leq t \right\};$$

(H10) the function $\int\limits_t^{t+h} |b(u)|du$ is bounded on R_+;

(H11) $\int\limits_{t-h}^{t} |\lambda(s)|ds \le \Lambda_0 = \text{const. for } t \ge 0;$

(H12) the function $|\lambda(t)|$ is bounded on R_+;

(H13) $\int\limits_{t-h}^{t} |\lambda(s)|ds \ge \lambda_0 > 0$ for $t \ge 0$ and λ_0 const.;

(H14) the function $\int\limits_{t}^{t+h} |b(u)|du\big[\int\limits_{t-h}^{t} \lambda^2(u)du\big]^{1/2}$ is bounded for $t \ge 0$;

(H15) the function $\int\limits_{0}^{t} |\lambda(u)| \int\limits_{u}^{u+h} |b(s)|ds\, du$ is uniformly continuous on R_+;

(H16) $\limsup\limits_{t\to\infty} \big[B(t) \int\limits_{t-h}^{t} |\lambda(s)|ds\big] > 0.$

The following relations can be easily proved (see Remark 4 and Theorem 11):

(a) (H4) does not imply (H5) which does imply (H4), while (H5) does not imply (H7) which does imply (H5);

(b) (H7) and (H12) imply (H6), but (H6) and (H12) do not imply (H7);

(c) (H9) implies (H8), but (H8) and (H10) do not imply (H9);

(d) (H10) and (H12) imply (H14);

(e) (H13) and ($b(t) \not\equiv 0$) imply (H16), but (H16) and (H10) do not imply (H13).

THEOREM 8.

(1) [(H2), (H3), (H9), and (H16)] imply that $x = 0$ is A.S.

(2) [(H1), (H3), (H9), and (H4)] imply that $x = 0$ is A.S.

(3) [(H2), (H3), (H8), (H10), (H11), and (H13)] imply that $x = 0$ is U.A.S.

(4) [(H1), (H7), and (H14)] imply that $x = 0$ is U.A.S.

(5) [(H1), (H6), (H10), (H12), and (H13)] imply that $x = 0$ is U.A.S.

(6) [(H1), (H5), and (H15)] imply that $x = 0$ is U.A.S.

PROOF. Define the Liapunov functional

$$V(t, x_t) = |x(t)| + \alpha \int_{-h}^{0} \int_{t+s}^{t} |\lambda(u)|\,|b(u-s)|\,|g(x(u))|du\,ds.$$

Since $|g(x)| \le c|f(x)|$, its derivative satisfies

$$V'(t,x_t) \le -\left[a(t) - \alpha c|\lambda(t)| \int_t^{t+h} |b(u)|du\right] |f(x(t))| - (\alpha-1)|b(t)| \int_{t-h}^t |\lambda(s)|\,|g(x(s))|ds.$$

Changing the order of integration yields the identity

$$\int_{-h}^0 \int_{t+s}^t |\lambda(u)|\,|b(u-s)||g(x(u))|du\,ds = \int_{t-h}^t |\lambda(u)| \int_t^{u+h} |b(v)|dv|g(x(u))|du \qquad (*)$$

from which we obtain

$$V(t,x_t) \le |x(t)| + \alpha \int_t^{t+h} |b(u)|du \int_{t-h}^t |\lambda(u)|\,|g(x(u))|du.$$

To prove (1), apply Theorem 2(A) and Remark 1 with $D(t,x_t) = |\lambda(t)g(x(t))|$. (The assertion in Example 1 can be obtained from Theorem 8(1) by taking $\lambda(t) \equiv 1$.)

To prove (2) and (3), apply Theorem 2(B) and Theorem 3, respectively, with the same $D(t,x_t) = |\lambda(t)g(x(t))|$.

To prove (4), put $\alpha = 1$ in the Liapunov functional $V(t,x_t)$. By the Schwarz inequality and $|g(x)| \le c|f(x)|$ we have

$$V(t,x_t) \le |x(t)| + \int_t^{t+h} |b(u)|du \left[\int_{t-h}^t |\lambda(u)|^2 du\right]^{1/2} c \left[\int_{t-h}^t f^2(x(u))du\right]^{1/2}.$$

The assertion follows from Theorem 5 with $D(t,x_t) = f^2(x(t))$.

To prove (5), let $\alpha = 1$ again and apply Theorem 5 with $D(t,x_t) = |\lambda(t)f(x(t))|$.

To prove (6), define

$$Z(t,x_t) = \int_{-h}^0 \int_t^{t+s} |\lambda(u)|\,|b(u-s)|\,|g(x(u))|du\,ds.$$

Then

$$Z'(t,x_t) \leqq |\lambda(t)| \int_{-h}^0 |b(t-s)|ds|g(x(t))| \leqq |\lambda(t)| \int_t^{t+h} |b(s)|ds|g(x(t))|.$$

Therefore, condition (iii) in Theorem 7 is met by (H15). On the other hand, according to the identity (*) we have

$$Z(t,\phi) \le \bar{g}(\|\phi\|) \int_{t-h}^t |\lambda(u)| \int_u^{u+h} |b(s)|ds\,du,$$

where

$$\bar{g}(v) := \max\{|g(s)| : |s| \le v\}, \quad v > 0.$$

Under (H15) the function $\int\limits_{t-h}^{t} |\lambda(u)| \int_u^{u+h} |b(s)| ds\, du$ is bounded on R_+, so all conditions of Theorem 7 are satisfied. This completes the proof.

It is known that when $f(x) = x$ and $\lambda(t) = b(t) = 0$ (i.e., for $x' = -a(t)x$) the condition $\int\limits_0^{\infty} a(t)dt = \infty$ is sufficient for A.S. This suggests the existence of such sufficient conditions for the asymptotic stability of the zero solutions of the equation in the title of the present section which would ask that λ and b be small and $\int\limits_{t-h}^{t} a(s)ds$ be large. Theorems 8(1) and (2) are not of this type (see (H9) and (H16)). However, such results can be derived from our theorems by choosing another Liapunov functional.

THEOREM 9. Suppose that $c > 0$ and

(i) $a(t) \ge 0$ for $t \in R_+$,

(ii) $|\lambda(t)| \le \gamma a(t)$ for $t \in R_+$ with some $\gamma > 0$, and

(iii) there is an $h_1 \in (0, h)$ such that the function $[1/c\gamma h] - |b(t)|$ is IP(h_1).

Then $x = 0$ is stable and $\lim\limits_{t\to\infty} x(t)$ exists and is finite.

If, in addition,

(iv) $f(x) \ne 0$ for $x \ne 0$ and

(v) $\limsup\limits_{t\to\infty} \int\limits_{t-h}^{t} a(s)ds > 0$

then $x = 0$ is A.S.

Finally, if (i), (ii), and (iv) hold together with

(iii′) there is an $h_1 \in (0, 1)$ such that $[1/c\gamma h] - |b(t)|$ is UWIP$(h_1, 4h)$ and

(v′) $0 < \alpha \le \int\limits_{t-h}^{t} a(u)du \le A$ for $t \in R_+$ and A constant,

then $x = 0$ is U.A.S.

PROOF. Consider the functional

$$V(t, x_t) = |x(t)| + (1/h) \int_{-h}^{0} \int_{t+s}^{t} a(u)|f(x(u))|du\, ds.$$

Its derivative satisfies

$$\begin{aligned} V'(t, x_t) &\leq |b(t)| \int_{t-h}^{t} |\lambda(s)|\, |g(x(s))|ds - (1/h) \int_{t-h}^{t} a(s)|f(x(s))|ds \\ &\leq [c\gamma|b(t)| - (1/h)] \int_{t-h}^{t} a(s)|f(x(s))|ds. \end{aligned}$$

Now the assertions follow from Theorems 2(A), (C), from Theorem 2(A) and Remark 1, and from Theorem 3, respectively, with $D(t, x_t) = a(t)|f(x(t))|$.

6 Appendix

The next theorem shows how one can relax conditions of Theorem 1(c) if only asymptotic stability is desired instead of U.A.S. It is not particularly useful but it is interesting because it is another result which fails when $h = 0$ and it fails in a different way than the previous such results fail when $h = 0$.

THEOREM 10. Let η be IP(h) and suppose that $V : R_+ \times C_H \to R_+$, is continuous and locally Lipschitz in ϕ with

(i) $W_1(|\phi(0)|) \leq V(t, \phi)$, $V(t, 0) = 0$ and

(ii) $V'_{(1)}(t, x_t) \leq -\eta(t)W_2(\|x_t\|)$.

Then $x = 0$ is A.S.

PROOF. We note that $x = 0$ is stable according to Theorem 1(a). For a given $t_0 \in R_+$ find $\delta > 0$ such that $\|\phi\| < \delta$ implies that $|x(t, t_0, \phi)| < H$ for $t \geq t_0$. Let $x(t) = x(t, t_0, \phi)$ and suppose that $x(t) \nrightarrow 0$ as $t \to \infty$. Then there is an $\epsilon > 0$ and a sequence $\{t_n\} \uparrow +\infty$ with $|x(t_n - h)| \geq \epsilon$ so that $\|x_t\| \geq \epsilon$ for $t_n - h \leq t \leq t_n$. We may pick a subsequence to ensure that $t_n + h < t_{n+1}$. Then for $t \geq t_n$ we have

$$V(t, x_t) \leq V(t_0, \phi) - \int_{t_0}^{t} \eta(s) W_2(\|x_s\|) ds \leq V(t_0, \phi) - \sum_{i=1}^{n} \int_{t_i - h}^{t_i} \eta(s) W_2(\|x_s\|) ds$$
$$\leq V(t_0, \phi) - \sum_{i=1}^{n} W_2(\epsilon) \int_{t_i - h}^{t_i} \eta(s) ds \to -\infty$$

as $n \to \infty$, a contradiction. This completes the proof.

We now consider the relation between the two main conditions on the coefficient functions η_i in our theorems.

THEOREM 11. Let $\eta : R_+ \to R_+$ be measurable.

(i) If η is positive in measure, then η is integrally positive.

(ii) If η is integrally positive, then η is not necessarily positive in measure.

PROOF. (i) Let η be PIM and let $I = \bigcup_{i=1}^{\infty} (a_i, b_i)$ with $b_i - a_i \geq \delta$ and $a_{i+1} \geq b_i$ for some $\delta > 0$ and all $i = 1, 2, \ldots$. We must show that $\int_t \eta(t) dt = \infty$. Now by definition of PIM there is a $\bar{\delta} > 0$ with $\int_{a_i}^{b_i} \eta(s) ds \geq \bar{\delta}$ for i sufficiently large. Hence, $\int_t \eta(t) dt = \infty$.

(ii) We construct a measurable function which is integrally positive but is not positive in measure. Let

$$\eta(t) = 0 \quad \text{if} \quad (m-1)h + (3i+1)(h/3^m) \leq t \leq (m-1)h + (3i+2)(h/3^m)$$
$$\text{for} \quad m = 1, 2, 3, \ldots$$

and for $i = 0, 1, \ldots, 3^{m-1} - 1$. Define $\eta(t) = 1$ otherwise. It is easy to see that for every $\delta > 0$ then $\lim_{t \to \infty} \int_{t-\delta}^{t} \eta(s) ds = 2\delta/3$. That is, η is integrally positive. On the other hand, let

$$Q_m = \{t \in ((m-1)h, mh) : \eta(t) = 0\}.$$

Then $Q_m \subset [mh - h, mh]$, it is open, $\mu(Q_m) = h/3$; nevertheless $\int_{Q_m} \eta = 0$, which means that η is not positive in measure. This completes the proof.

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