KRASNOSELSKII'S INVERSION PRINCIPLE AND FIXED POINTS

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Key words and phrases: Integral equations, implicit functions, fixed points.

1. INTRODUCTION

Krasnoselskii studied a paper of Schauder [16] and concluded (cf. Smart [18;p.31]):

PRINCIPLE: The inversion of a perturbed differential operator may yield the sum of a contraction and a compact operator.

The inversion is then some kind of an integral equation and the solution of that equation calls for a fixed point of a mapping. Krasnoselskii offered the following theorm as a prototype. Concerning the terminology of compact mapping used in this theorem, Krasnoselskii is using the convention of Smart [18;p. 25] to mean the following: Let A map a set M into a topological space X. If AM is contained in a compact subset of X, we say that A is compact. In particular, M need not be bounded. However, when we use the term elsewhere, we mean the standard term which says that bounded sets are mapped into compact sets.

THEOREM 1. (Krasnoselskii[13]) Let M be a closed convex nonempty subset of a Banach space $(\mathcal{B}, \|\cdot\|)$. Suppose that A and B map M into $\mathcal{B}$ such that

(i) $x, y \in M \Rightarrow Ax + By \in M$,

(ii) A is compact and continuous,

(iii) B is a contraction mapping.

Then $\exists\ y \in M$ with $y = Ay + By$.

When A is the zero operator, this is Banach's fixed point theorem; when B is zero, this is Schauder's second fixed point theorem.

We have studied that principle in four papers ([2-5]) and find it to be valid in a wide setting. The following five problems are typical examples. The separation of operators is clearly seen in neutral integral equations which may arise from neutral functional differential equations which have been discussed extensively in [9] and [14], for example. In the space allowed here we can offer few details, but those can be found in the aforementioned papers. In particular, we obtain three fixed point theorems parallel to that of Krasnoselskii.

PROBLEM 1. Given $f(t,x)$ in C^1 with $f(0,0) = 0$ and $\frac{\partial f}{\partial x}(0,0) \neq 0$, invert

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$$\frac{\partial f}{\partial t}(t,x) + \frac{\partial f(t,x)}{\partial x}\frac{dx}{dt} = 0 \tag{1}$$

and find the implicit function φ with $f(t,\varphi(t)) = 0$, $\varphi(0) = 0$. This problem has two well-known solutions; Problem 5 extends it.

PROBLEM 2. Let $a(t+T) = a(t)$, $g(t+T,x) = g(t,x)$, $p(t+T) = p(t)$, $e^{\int_0^T -a(s)ds} = \alpha < 1$. Invert

$$x' = -a(t)x - g(t,x) \tag{2}$$

and find a T-periodic solution.

PROBLEM 3. Let $|\alpha| < 1$, $a > 0$, $h > 0$, q be continuous. Invert

$$x' = \alpha x'(t-h) + ax - q(t,x,x(t-h)) \tag{3}$$

and find a solution for a given continuous initial function φ.

PROBLEM 4. Let $|\alpha| < 1$, $a > 0$, $h > 0$, $p(t+T) = p(t)$, $g(t+T,x,y) = g(t,x,y)$. Invert

$$x' = \alpha x'(t-h) + ax - g(t,x,x(t-h)) \tag{4}$$

and find a T-periodic solution.

PROBLEM 5. Given $f(t,x)$ in C^2 with $f(0,0) = 0$, $\frac{\partial f}{\partial x}(t,x) > 0$ for (t,x) near $(0,0)$ and $(t,x) \neq (0,0)$. Invert

$$\frac{\partial f}{\partial t}(t,x) + \frac{\partial f(t,x)}{\partial x}\frac{dx}{dt} = 0 \tag{5}$$

and find φ with $\varphi(0) = 0$ and $f(t,\varphi(t)) = 0$.

2. SOLUTION OF PROBLEM 1

This is an interesting problem in view of Krasnoselskii's theorem. It can be inverted using contraction mappings (cf. Smart [18:p.6]) for a unique solution. On the other hand, Hartman [11;p.10] notes that since $\frac{\partial f}{\partial x}(0,0) \neq 0$, we can divide by that quantity and obtain a differential equation, say $x' = G(t,x)$, with G continuous so that the inversion is

$$x(t) = \int_0^t G(s,x(s)ds;$$

and this will define a compact mapping of an appropriate set. There is a solution (not necessarily unique!) by Schauder's theorem. The details of finding M are given in Smart [18;pp.44-5].

3. SOLUTION OF PROBLEM 2

Write (2) as

$$\left(x \exp \int_0^t a(s)\,ds\right)' = -g(t,x) \exp \int_0^t a(s)\,ds$$

and integrate from $t-T$ to t obtaining

$$x(t) = x(t-T)\exp - \int_{t-T}^{t} a(s)\,ds \tag{2*}$$
$$- \int_{t-T}^{t} g(u,x(u))\left[\exp - \int_{u}^{t} a(s)\,ds\right] du.$$

If $\exp - \int_{t-T}^{t} a(s)\,ds = \alpha < 1$ and if $(\mathcal{B}, \|\cdot\|)$ is the Banach space of continuous T-periodic functions $\varphi : R \to R$, then (2*) can be expressed as

$$\varphi(t) = (B\varphi)(t) + (A\varphi)(t)$$

where B is a contraction and A maps bounded subsets of $\mathcal{B}$ into compact subsets of $\mathcal{B}$. In fact, B can take a portion of the integral which might not be well-behaved in some sense. The above integral maps bounded sets of T-periodic functions into equicontinuous sets.

Now (2*) is a typical example of

$$x(t) = f(t, x(t-T) + \int_{t-T}^{t} g(u,x(u))D(t,u)du \tag{(2**)}$$

where f defines a contraction operator on a space of periodic functions, while g and D are continuous. If g is small enough, then Krasnoselskii's theorem readily applies. But if g is not small then a different type of theorem is needed.

Krasnoselskii's idea was to combine the Banach and Schauder fixed point theorems. In [5] we sought to extend his idea by combining a result of Schaefer [15] on fixed points from a priori bounds with Banach's theorem. Schaefer's result is for locally convex topological vector spaces, but the formulation here is that of Smart [18;p.29].

THEOREM 2. (Schaefer[15]) Let $(\mathcal{B}, \|\cdot\|)$ be a normed space, H a continuous mapping of $\mathcal{B}$ into $\mathcal{B}$ which is compact on each bounded subset X of $\mathcal{B}$. Then either

(i) the equation $x = \lambda Hx$ has a solution for $\lambda = 1$, or

(ii) the set of all such solutions x, for $0 < \lambda < 1$, is unbounded.

Our generalization may be stated as follows.

THEOREM 3. (Burton-Kirk[5]) Let $(\mathcal{B}, \|\cdot\|)$ be a Banach space, A, $B : \mathcal{B} \to \mathcal{B}$, B a contraction with contraction constant $\alpha < 1$, and A continuous with A mapping bounded sets into compact sets. Either

(i) $x = \lambda B(x/\lambda) + \lambda Ax$ has a solution in $\mathcal{B}$ for $\lambda = 1$, or

(ii) the set of all such solutions, $0 < \lambda < 1$, is unbounded.

With that result we can obtain a strong conclusion for (2**) as follows.

Let $(\mathcal{B}, \|\cdot\|)$ be the Banach space of continuous T-periodic functions $\varphi : R \to R$ with the supremum norm. Consider (2**) and suppose there is a $T > 0$ and $\alpha \in (0,1)$ with:

$$f(t+T,x) = f(t,x), \quad D(t+T,s+T) = D(t,s), \quad g(t+T,x) = g(t,x),$$
$$t-h \le s \le t \text{ implies that } D_s(t,t-h) \ge 0,$$
$$D_{st}(t,s) \le 0, \quad D(t,t-h) = 0,$$
$$|f(t,x) - f(t,y)| \le \alpha|x-y|, \quad xg(t,x) \ge 0,$$
$$\forall\, k > 0\, \exists\, P > 0\, \exists\, \beta > 0 \text{ with}$$
$$2\lambda[-(1-\alpha)xg(t,x) + k|g(t,x)|\,] \le \lambda[P - \beta|g(t,x)|\,],$$
$$f, g, \text{ and } D_{st} \text{ are continuous.}$$

THEOREM 4. ([5]) Under these conditions, (2^{**}) has a T-periodic solution.

In proving this theorem we needed to establish a priori bounds for the homotopy equation

$$x(t) = \lambda f\left(t, \frac{x}{\lambda}\right) - \lambda \int_{t-h}^{t} D(t,s)g(s,x(s))\,ds. \tag{2^{**}_λ}$$

This is nontrivial and we give here a brief outline of how it is done. Full details are found in [5].

Suppose that $x \in \mathcal{B}$ is a solution. Define

$$V(t) = \lambda^2 \int_{t-h}^{t} D_s(t,s)\left(\int_s^t g(v,x(v))\,dv\right)^2 ds.$$

This is a type of Liapunov function obtained from (2^{**}_λ) by squaring $x - \lambda f$, integrating by parts, and using the Schwarz inequality.

Now $D_{st}(t,s) \le 0$ so

$$\begin{aligned} V'(t) &\le -\lambda^2 D_s(t,t-h)\left(\int_{t-h}^{t} g(v,x(v))\,dv\right)^2 \\ &+2\lambda^2 g(t,x)\int_{t-h}^{t} D_s(t,s)\int_s^t g(v,x(v))\,dv\,ds. \end{aligned}$$

Working with this, (2^{**}_λ), and integrating by parts we obtain

$$\begin{aligned} V'(t) &\le 2\lambda\{|g(t,x)|[\alpha|x|+k] - xg(t,x)\} \\ &= 2\lambda[|\alpha x g(t,x)| + k|g(t,x)| - xg(t,x)] \\ &\le 2\lambda[-(1-\alpha)xg(t,x) + k|g(t,x)|\,] \end{aligned}$$

for some $k > 0$. As $\alpha < 1$, we find $\beta > 0$ and $P > 0$ with

$$V'(t) \le \lambda[-\beta|g(t,x)| + P].$$

Thus, $x \in \mathcal{B}$ implies $V \in \mathcal{B}$ so that

$$0 = V(T) - V(0) \le \lambda\left[-\beta\int_0^T |g(t,x(t))|\,dt + PT\right]$$

or

$$\int_0^T |g(t,x(t))|\,dt \le PT/\beta$$

since $\lambda > 0$. As $g(t,x(t)) \in \mathcal{B}$, there is an $n > 0$ with

$$\int_{t-h}^{t} |g(t,x(t))|\,dt \le n.$$

Taking $M = \max\limits_{-h \le s \le t \le T} |D(t,s)|$ we finally have

$$\begin{aligned} |x(t)| &\le \left|\lambda f\left(t, \frac{x}{\lambda}\right)\right| + \lambda\left|\int_{t-h}^{t} D(t,s)g(s,x(s))\,ds\right| \\ &\le \alpha|x(t)| + k + Mn \end{aligned}$$

or

$$\|x\| \le (Mn + k)/(1-\alpha),$$

a suitable a priori bound.

The other conditions are routinely established.

4. SOLUTION OF PROBLEM 3

Write (3) as

$$(x - \alpha x(t-h))' = a(x - \alpha x(t-h)) + a\alpha x(t-h) - q(t, x, x(t-h)),$$

multiply by e^{-at}, and group terms as

$$\big[(x - \alpha x(t-h))e^{-at}\big]' = \big[a\alpha x(t-h) - q(t, x, x(t-h))\big]e^{-at}.$$

We search for a solution having the property that

$$(x(t) - \alpha x(t-h))e^{-at} \to 0 \text{ as } t \to \infty$$

so that an integration from t to infinity yields

$$-(x(t) - \alpha x(t-h))e^{-at} = \int_t^\infty \big[a\alpha x(s-h) - q(s, x(s), x(s-h))\big]e^{-as}ds$$

and, finally

$$x(t) = \alpha x(t-h) + \int_t^\infty \big[q(s, x(s), x(s-h)) - a\alpha x(s-h)\big]e^{a(t-s)}ds. \tag{3*}$$

A general form for such equations is

$$x(t) = f(x(t-h)) + \int_t^\infty Q(s, x(s), x(s-h))C(t-s)ds + p(t). \tag{3**}$$

We call this a neutral delay integral equation of advanced type and it is a very interesting equation. It may have a solution on all of R or it may have a solution on $[0,\infty)$ generated by an initial function φ on $[-h, 0]$. In the latter case, notice that we can not obtain a local solution: we must get the full solution on $[0,\infty)$. Thus, we will need to employ a fixed point theorem to get existence and that means that we will get a fixed point in the solution space; hence, we must know in advance the form of the solution space. We study this problem in [2] and find that the solution will have discontinuities at $t = nh$, but the jumps will tend to zero as $n \to \infty$. (This is parallel to solutions of functional differential equations smoothing (cf. El'sgol'ts [8]).)Equally important is the need to know in advance the growth of the solution so that functions in the solution space will have a weighted norm allowing such growth. We illustrate this in Problem 3 using only contraction mappings for simplicity. In Remark 3 we tell how the problem can be solved using Krasnoselskii's result without the Lipschitz condition on Q. In Problem 4 when we are seeking periodic solutions we will use the sum of a contraction and compact map.

Thus, for (3^{**}) we will suppose that

$$|f(x) - f(y)| \le \alpha|x - y|, \quad 0 \le \alpha < 1,$$

and for $0 \le k \le 1$

$$\begin{aligned}
&|Q(t,x,y) - Q(t,w,z)| \le (k|x-w| + (1-k)|y-z|),\\
&|Q(t,0,0)| \le 1,\\
&Q,\ p,\ \text{and } C \text{ are continuous,}\ \int_0^\infty |C(-u)|du =: C_0 < \infty.
\end{aligned}$$

Suppose that there is a given initial function $\psi : [-h, 0] \to R$ which is continuous. The initial value problem for (3^{**}) asks that we find a function $x : [-h, \infty) \to R$, denoted by $x(t, 0, \psi)$, with $x(t, 0, \psi) = \psi(t)$ on $[-h, 0)$ and x satisfies (2) on $[0, \infty)$.

REMARK 1. If $x(t) = x(t, 0, \psi)$ is to be continuous at $t = 0$, then we must have

$$x(0) = \psi(0) = f(\psi(-h)) + \int_0^\infty Q(s, x(s), x(s-h))C(-s)ds + p(0).$$

Thus, for an arbitrary continuous ψ, a finite jump discontinuity at $t = 0$ must be expected. Under conditions to be given, the integral in (3^{**}) will be continuous whenever x is piecewise continuous and so, from (3^{**}), we must expect x to have jump discontinuities at nh, for $n = 1, 2, \dots$ This shows us how to search for a solution on $[0, \infty)$.

First, examine $p(t)$ and select a constant $K > 0$ and a continuous function $D : [-h, \infty) \to [1, \infty)$ with

$$\sup_{t \ge -h} |p(t)/D(t)| < \infty \text{ and } D(t-h)/D(t) \le K.$$

The function D will be the weight for a norm on a Banach space.

Next, for D to be compatible with $C(t-s)$ we will need

$$\sup_{t \ge 0} \int_t^\infty |C(t-s)|[D(s)/D(t)]ds < \infty.$$

Finally, we will want a mapping induced by (3^{**}) to map piecewise continuous functions into piecewise continuous functions. This will lead us to ask that

$$\int_t^\infty Q(s, \varphi(s), \varphi(s-h))C(t-s)ds$$

be continuous whenever φ is piecewise continuous and $\varphi(t)/D(t)$ is bounded. It may be noted that if φ is bounded and piecewise continuous then a classical theorem ([12;p.398]) says that this integral is uniformly continuous. We ask that

$$\begin{cases} \forall J > 0,\ \text{if } 0 \le t_1 < t_2 \le J,\ \text{then} \\ \displaystyle\int_{t_2}^\infty |C(t_1 - s) - C(t_2 - s)|D(s)ds \to 0 \text{ as } |t_1 - t_2| \to 0. \end{cases}$$

In order to have a contraction mapping we also need

$$\alpha K + (k + (1-k)K) \sup_{t \ge 0} \int_t^\infty |C(t-s)|[D(s)/D(t)]ds =: \mu < 1.$$

Define

$$(\mathcal{B}, |\cdot|_D)$$

as the Banach space of functions $\varphi : [-h, \infty) \to R$ which are continuous on $[(n-1)h, nh)$ with left-hand limits existing at nh and with the property that

$$|\varphi|_D := \sup_{t \geq -h} |\varphi(t)/D(t)| \text{ exists.}$$

THEOREM 5. ([2]) Let the stated conditions hold and let $\psi : [-h, 0] \to R$ be a given continuous function. Then there is a unique $x \in \mathcal{B}$ satisfying (3^{**}) for $t \geq 0$ and $x(t) = \psi(t)$ on $[-h, 0)$.

The proof is a straightforward contraction mapping argument.

REMARK 2. If $p(t+T) = p(t)$ and $Q(t+T, x, y) = Q(t, x, y)$ then we can take $D(t) = 1$ and prove that under the same conditions there is a unique periodic solution.

REMARK 3. (Exercise) If $D(t) \to \infty$ as $t \to \infty$, then the Lipschitz condition on Q can be replaced by $|Q(t, x, y)| \leq (1 + k|x| + (1-k)y|)$ and Krasnoselskii's result can be used. The number μ is defined in the same way and the set M is readily found. The details of showing that bounded sets are mapped into compact sets is similar to work found in Burton [1;p.170].

5. FIRST SOLUTION OF PROBLEM 4

We specialize (4) to

$$x(t) = \alpha x(t-h) + \int_t^\infty (\beta x^{2n+1}(s) + \gamma x^m(s-h))C(t,s)ds + p(t) \tag{4^*}$$

and write the corresponding homotopy equation as

$$x(t) = \alpha x(t-h) + \lambda\left[\int_t^\infty (\beta x^{2n+1}(s) + \gamma x^m(s-h))C(t,s)ds + p(t)\right] \tag{4^*_λ}$$

in which

$$\begin{gathered}|f(x) - f(y)| \leq \alpha|x-y|,\ 0 \leq \alpha < 1,\ xg(x) > 0 \text{ if } x \neq 0,\\ g, r, C, C_s, C_{st}, p \text{ are continuous, } p(t+T) = p(t),\\ C(t+T, s+T) = C(t,s),\end{gathered}$$

and

$$\begin{gathered}\sup_{t \geq 0}\left[\int_t^\infty |C(t,s)|ds + \int_t^\infty (|C_s(t,s)| + |C_{st}(t,s)|)(t-s)^2 ds\right] < \infty,\\ |C(t,s)||t-s| \to 0 \text{ as } s \to \infty.\end{gathered}$$

THEOREM 6. ([2]) If $C_{st} \leq 0$, $|\alpha| \leq 1$, $\beta < 0$, if either $2n > m$ or if $[\gamma\alpha > 0$, $2n+2 = m+1$, and $|\beta| > |\gamma|]$, then (4^*) has a T-periodic solution.

The proof is based on Theorem 3. Complete details are given in [5], but we give a brief sketch here of just how the a prior bounds are obtained.

Define a Liapunov functional by

$$V(t)=\int_t^\infty \lambda^2 C_s(t,s)\Big(\int_s^t [\beta x^{2n+1}x(u)+\gamma x^m(u-h)]\,du\Big)^2 ds.$$

Our space is $\mathcal{P}_T$ of continuous T-periodic functions with the supremem norm. If $x\in\mathcal{P}_T$ is a solution, then the derivative of this function satisfies

$$\begin{aligned}V'(t)=\lambda^2\int_t^\infty C_{st}(t,s)\Big(\int_s^t [\beta x^{2n+1}(u)+\gamma x^m(u-h)]du\Big)^2 ds\\ +2\lambda(\beta x^{2n+1}+\gamma x^m(t-h))(x-\alpha x(t-h)-\lambda p(t)).\end{aligned}$$

Next, define the function

$$\begin{aligned}W(t)=2\lambda\Big[\big(\frac{|\gamma|m}{m+1}+|\gamma\alpha|\big)\int_{t-h}^t |x(s)|^{m+1}ds\\ +|\gamma\lambda|\,\|\varphi\|\int_{t-h}^t |x(s)|^m ds+\frac{|\alpha\beta|}{2n+2}\int_{t-h}^t x^{2n+2}(s)ds\Big].\end{aligned}$$

with derivative

$$\begin{aligned}W'(t)=2\lambda\Big[\big(\frac{|\gamma|m}{m+1}+|\gamma\alpha|\big)(|x|^{m+1}-|x(t-h)|^{m+1}\\ +|\gamma\lambda|\,\|\varphi\|\,(|x|^m-|x(t-h)|^m)+\frac{|\alpha\beta|}{2n+2}(x^{2n+2}-x^{2n+2}(t-h))\Big].\end{aligned}$$

Now form $V+W$. Since $m<2n$, and since $\beta<0$ and $|\alpha|<1$, there is an $M>0$ with

$$(V+W)'\le 2\lambda\left[-\frac{|\beta|(1-|\alpha|)}{2}x^{2n+2}+M\right].$$

But $x\in\mathcal{P}_T$ yields $V+W\in\mathcal{P}_T$ so

$$\begin{aligned}0&=V(T)+W(T)-V(0)-W(0)\\ &\le 2\lambda\Big[-\frac{|\beta|(1-|\alpha|)}{2}\int_0^T x^{2n+2}(s)ds+MT\Big].\end{aligned}$$

As $\lambda>0$ we find $X>0$ with

$$\int_0^T x^{2n+2}(s)ds\le X.$$

With the aid of (4^*_λ), this yields a bound on the supremum norm of the solution.

6. SECOND SOLUTION OF PROBLEM 4

In order to get an intuitive idea of the problem, consider again (3^*) and the special case

$$x(t)=\int_t^\infty \beta x(s)e^{a(t-s)}ds.$$

If we reduce this to an ordinary differential equation and take $\beta < 0$, as in our last result, we see that this is the unstable case. It turns out that the case of $\beta > 0$ is much more challenging and interesting. The following simple case illustrates the difficulties.

The equation of interest is

$$x(t) = \alpha x(t-h) + \beta \int_t^\infty x(s) e^{a(t-s)} ds + p(t)$$

with p continuous, $p(t+T) = p(t)$, $|\alpha| < 1$, $a > 0$, and $\beta > 0$.

We attempt to use Theorem 3 by placing λ before the last two terms and construct a Liapunov function in order to get an a priori bound. The bound fails as λ tends to zero. The next attempt is with

$$x(t) = \alpha x(t-h) + \lambda \left[\int_t^\infty \beta \phi(s) e^{\lambda a(t-s)} ds + p(t) \right]$$

The a priori bound now holds, but continuity is lost as $\lambda \to 0$. Moreover, since λ no longer enters linearly, it is clear that we will need to prove a new fixed point theorem. To keep continuity we note that if we are looking for a T-periodic solution of (3*), we can telescope the equation to

$$x(t) = \alpha x(t-h) + (\beta/\{1 - e^{-aT}\}) \int_t^{t+T} x(s) e^{a(t-s)} ds$$

the sum of a contraction and compact map. Our homotopy equation will be

$$(P\varphi)(t) = \alpha \varphi(t-h) + \{\lambda\beta/[1 - e^{-\lambda a T}]\} \int_t^{t+T} \varphi(u) e^{\lambda a(t-u)} du + \lambda p(t).$$

THEOREM 7. If $|\alpha| < 1$, $a > 0$, $\beta > 0$, and if

$$\beta > a + a\alpha^2 + |\alpha(\beta - 2a)|,$$

then there is a T-periodic solution.

Theorem 3 will not work because λ does not enter linearly. We need to replace Schaefer's theorem by the following result gleaned from Granas [10] and Eloe and Henderson [7].

THEOREM 8. (Granas-Eloe-Henderson [10 and 7]) Let $(\mathcal{S}, \|\cdot\|)$ be a Banach space, $K > 0$, $X = \{\varphi \in \mathcal{S} \,\Big|\, \|\varphi\| \le K\}$, and $A = \{\varphi \in X \,\Big|\, \|\varphi\| = K\}$. Suppose that $h : [0,1] \times X \to \mathcal{S}$ is continuous and for each fixed $\lambda_0 \in [0,1]$, then $h(\lambda_0, X)$ has compact closure. Moreover, suppose that $h(\lambda, \varphi) = \varphi$ has no solution $\varphi \in A$ and that there is a $\psi \in X \backslash A$ with $h(0, \varphi) = \psi$ for all $\varphi \in X$. Then $h(1, p) = p$ has a solution in X.

We combine this result with Banach's theorem and obtain the following theorem and corollary which will solve our problem.

THEOREM 9 (Burton [4]) Let $(\mathcal{S}, \|\cdot\|)$ be a Banach space and suppose that:

(i) $B : \mathcal{S} \to \mathcal{S}$ is a contraction (so $(I - B)^{-1}$ is continuous);

(ii) $h : (0,1] \times \mathcal{S} \to \mathcal{S}$, $\lim_{\lambda \to 0^+} \lambda(I-B)^{-1}h(\lambda,x) =: D(x)$ exists for each $x \in \mathcal{S}$;

(iii) $H(\lambda,x) =: \begin{cases} \lambda(I-B)^{-1}h(\lambda,x) & \text{if } \lambda \neq 0 \\ D(x) & \text{if } \lambda = 0 \end{cases}$
is jointly continuous in (λ, x) and for fixed λ maps bounded sets into sets with compact closure;

(iv) there is a $K > 0$ such that any solution of

$$x = \lambda B(x/\lambda) + \lambda h(\lambda,x), \quad 0 < \lambda \leq 1, \tag{$*$}$$

satisfies $\|x\| \leq K$;

(v) $H(0,x) = 0$ for each $x \in \mathcal{S}$.

Under these conditions, (*) has a solution for $\lambda = 1$.

COR. Let $(\mathcal{S}, \|\cdot\|)$ be a Banach space and suppose that:

(i) $B : \mathcal{S} \to \mathcal{S}$ is a contraction;

(ii) $h : [0,1] \times \mathcal{S} \to \mathcal{S}$ is jointly continuous in (λ, x) and for fixed λ maps bounded sets into sets with compact closure;

(iii) there is a constant $K > 0$ such that any solution of

$$x = \lambda B(x/\lambda) + \lambda h(\lambda,x), \quad 0 < \lambda \leq 1, \tag{$*$}$$

satisfies $\|x\| \leq K$;

Under these conditions, (*) has a solution for $\lambda = 1$.

Because our B is linear, the corollary applies. Moreover, we work in the space of T-periodic functions with mean value zero so that when λ is zero, our space will be mapped onto the zero function. The main problem then is to find a priori bounds. We do this by defining

$$V(t) = \left(\lambda \overline{\beta} \int_t^{t+T} x(u) e^{\lambda a(t-u)} du \right)^2,$$

together with

$$W(t) = 2\lambda\big((1/2)|\alpha(\beta - 2a)| + a\alpha^2 + \epsilon_1\big) \int_{t-h}^{t} x^2(s) ds$$

so that

$$(V(t) + W(t))' \leq 2\lambda \left[(-\beta t + a + \epsilon_2 + |\alpha(\beta - 2a)| + a\alpha^2 + \epsilon_1)x^2 + \lambda M_4 \|p\| \ \right]$$

or

$$(V(t) + W(t))' \leq \lambda[-\mu x^2(t) + M]$$

for some $\mu > 0$ and $M > 0$, by the assumptions. This is then parlayed into a bound on the supremum norm as we have done before.

7. SOLUTION OF PROBLEM 5

To this point we have focused on Krasnoselskii's theorem by changing the compact map of a fixed set into a compact map with a priori bound. Now we consider the problem of weakening the contraction mapping.

Write (5) as

$$P(t,x)x' = F(t,x), \quad P(0,0) = 0,$$

We can partially invert that as

$$\int_0^x P(t,s)\,ds = \int_0^t \left[F(s,x(s)) + \int_0^{x(s)} P_t(s,v)\,dv\right] ds.$$

Write this as

$$V(t,x) = S\left(t, \int_0^t H(t,s,x(s))\,ds\right)$$

noticing that $S(0,0) = V(t,0) = 0$. We think of this as an operator equation $\tilde{V}\varphi = A\varphi$. A fixed point of $(P\varphi)(t) = (A\varphi)(t) - (\tilde{V}\varphi)(t) + \varphi(t)$ will solve our equation. Thus, we have a compact mapping A and a mapping $B = I - \tilde{V}$. Now B is not generally a contraction if $\partial V(0,0)/\partial x = 0$; but if $\partial V/\partial x > 0$ for $(t,x) \neq (0,0)$, then B is a shrinking map. (If $\partial V/\partial x < 0$ for $(t,x) \neq (0,0)$, then use $P\varphi = -A\varphi + \tilde{V}\varphi + I\varphi$.) This is what we will call a large contraction. Many other kinds of shrinking maps might possibly be substituted for the following, and a good selection is found in Sine [17].

DEFINITION. Let (M,ρ) be a metric space and $B : M \to M$. B is said to be a *large contraction* if $\varphi, \psi \in M$, with $\varphi \neq \psi$ then $\rho(B\varphi, B\psi) < \rho(\varphi,\psi)$ and if $\forall \varepsilon > 0\ \exists \delta < 1$ such that $[\varphi,\psi \in M, \rho(\varphi,\psi) \geq \varepsilon] \Rightarrow \rho(B\varphi, B\psi) \leq \delta\rho(\varphi,\psi)$.

The next three results are found in [3].

THEOREM 10. Let (M,ρ) be a complete metric space and B be a large contraction. Suppose there is an $x \in M$ and an $L > 0$, such that $\rho(x, B^n x) \leq L$ for all $n \geq 1$. Then B has a unique fixed point in M.

The following theorem will extend Krasnoselskii's result by weakening the requirement for a contraction and it will allow us to provide one solution to Problem 5.

THEOREM 11. Let $(S, \|\cdot\|)$ be a Banach space, M a bounded, convex nonempty subset of S. Suppose that A, $B : M \to M$ and that

$$x, y \in M \Rightarrow Ax + By \in M, \tag{i}$$

$$A \text{ is continuous and } AM \text{ is contained in a compact subset of } M, \tag{ii}$$

$$B \text{ is a large contraction.} \tag{iii}$$

Then $\exists\ y \in M$ with $Ay + By = y$.

THEOREM 12. Let $\alpha > 0$, V, $S : [-\alpha,\alpha] \times [-\alpha,\alpha] \to R$, $H : [-\alpha,\alpha] \times [-\alpha,\alpha] \times [-\alpha,\alpha] \to R$ be continuous, $S(0,0) = V(t,0) = 0$. Suppose that $(X, \|\cdot\|)$ is the Banach space of continuous $\varphi : [-\beta,\beta] \to R$, with the supremum norm, $0 < \beta < \alpha$, $M = \{\varphi \in X | \varphi(0) = 0, \|\varphi\| \leq \alpha\}$, and that $(B\varphi)(t) := \varphi(t) - V(t,\varphi(t))$ defines a large contraction on M. Then (5) has a solution in M.

REFERENCES

1. BURTON, T. A., *Stability and Periodic Solutions of Ordinary and Functional Differential Equations*, Academic Press, Orlando (1985).

2. BURTON, T. A., Basic neutral integral equations of advanced type, *J. Nonlinear Analysis*, to appear.

3. BURTON, T. A., Integral equations, implicit functions, and fixed points, *Proc. Amer. Math. Soc.*, to appear.

4. BURTON, T. A., Neutral integral equations and a fixed point theorem of Granas and Krasnoselskii type, submitted.

5. BURTON, T. A. and KIRK, COLLEEN, A fixed point theorem of Krasnoselskii-Schaefer type, submitted.

6. CORDUNEANU, C., *Integral Equations and Applications*, Cambridge Univ. Press, Cambridge (1991).

7 ELOE, P. W. and HENDERSON, J., Nonlinear boundary value problems and a priori bounds on solutions, *SIAM J. Math. Anal.* **15.** 642-647 (1984).

8. EL'SGOL'TS, L.E., *Introduction to the Theory of Differential Equations with Deviating Arguments*, Holden-Day, San Francisco (1966).

9. GOPALSAMY, K., *Stability and Oscillations in Delay Differential Equations of Population Dynamics*, Kluwer, Dordrecht (1992).

10. GRANAS, A., Sur la méthode de continuité de Poincaré, *C.R. Acad. Sc. Paris*, **t. 282.** 983–985 (1976).

11. HARTMAN, PHILIP, *Ordinary Differential Equations*, Wiley, New York (1973).

12. HEWITT, E. and STROMBERG, K., *Real and Abstract Analysis*, Springer, Berlin (1971).

13. KRASNOSELSKII, M.A., in *Amer. Math. Soc. Transl.* (2) **10.** 345–409 (1958).

14. KUANG, YANG, *Delay Differential Equations with Applications to Population Dynamics*, Academic Press, Boston (1993).

15. SCHAEFER, H., Über die Methode der a priori Schranken, *Math. Ann.* **129**(1955), 415–416.

16. SCHAUDER, J., Über den Zusammenhang zwischen der Eindeutigkeit and Lösbarkeit partieller Differentialgleichungen zweiter Ordung von Elliptischen Typus, *Math. Ann.* (1932), 661–721.

17. SINE, Robert C., *Fixed Points and Nonexpansive Mappings,* Amer. Math. Soc. (Contemporary Mathematics Vol. 18), Providence (1983).

18. SMART, D.R., *Fixed Point Theorems*, Cambridge Univ. Press, Cambridge (1980).