PERIODIC SOLUTIONS OF A NEUTRAL INTEGRO-DIFFERENTIAL EQUATION

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1. Introduction. This paper is devoted to the study of a neutral functional differential equation of the form

$$x'(t) = u(t, x'(\cdot)) + \int_{-\infty}^{t} f(t, s, x(s))\, ds + \int_{t}^{\infty} g(t, s, x(s))\, ds + p(t) \tag{1}$$

in which u, f, g, and p are periodic in a sense to be described later. The object is to give conditions to ensure that (1) has a periodic solution. It is supposed that u defines a contraction mapping, while the integral terms define compact mappings so that an application of Krasnoselskii's fixed point theorem yields the desired periodic solution.

The novelty here is that we use the right-hand-side of (1) itself to define the fixed point mapping rather than convert (1) to an integral equation which would be difficult to treat when $u \neq 0$.

Both integral and integrodifferential equations having a right-hand-side of the form of (1) occur frequently in the study of unstable manifolds. An example of an integral equation with $u = 0$ is found in Coddington and Levinson [2; pp. 330–332] and it is of arbitrary dimensions. A neutral predator-prey system is readily converted to an integral equation of this type.

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But one of the most interesting problems occurs in the study of the common equation

$$x'' - x = g(t, x, x', x'').$$

When we invert that differential operator we obtain two scalar equations of the form of (1); one is an integral equation, the other is an integrodifferential equation. Seeing the details is a very worthwhile motivational study and we devote Section 2 to it.

2. An example. The scalar equation $x'' - x = \sin t$ has a general solution $x(t) = c_1 e^t + c_2 e^{-t} - \frac{1}{2}\sin t$; there is a periodic solution.

Equation (1) is a neutral equation. Neutral equations may have delays in the derivative of a solution, as well as in the solutions (cf. Hale [2; pp. 24–32], Lakshmikantham et al [2]). Thus, a simple example is

$$(E1) \qquad x''(t) - x(t) = g(t, x(t), x'(t-h), x''(t-h))$$

where h is a positive constant. This equation is readily transformed into (1) and is then well suited to application of fixed point theory. With suitable restrictions it will also have a periodic solution. In particular, we will need $T > 0$ with $g(t+T, x, y, z) = g(t, x, y, z)$ and

$$g(t, x, y, z) = p(t) + r(t, x) + \beta y + \gamma z$$

where $|\beta - \gamma| + 2|\gamma| + |\beta + \gamma| < 2$, $\int_0^T p(s)\, ds = 0$, and r to be monotone, as described later.

To prove that (E1) has a T-periodic solution we set up a mapping with a fixed point φ in the Banach space of continuous T-periodic functions $(P_T, \|\cdot\|)$ with the supremum norm. The first theorem gives us two choices for mappings.

Theorem 1. *Let*

$$(E2) \qquad x(t) = -\frac{1}{2}\Big[\int_t^\infty e^{t-s} g(s, x(s), x'(s-h), x''(s-h))\, ds + \int_{-\infty}^t e^{-(t-s)} g(s, x(s), x'(s-h), x''(s-h))\, ds\Big]$$

and

$$(E3) \qquad x'(t) = \tfrac{1}{2}\Big[\textstyle\int_{-\infty}^{t} e^{-(t-s)} g(s, x(s), x'(s-h), x''(s-h))\, ds$$
$$- \int_t^{\infty} e^{t-s} g(s, x(s), x'(s-h), x''(s-h))\, ds\Big].$$

If $\varphi \in P_T$ satisfies either (E1), (E2), or (E3), then φ also satisfies the other two. Moreover, if $\varphi \in P_T$ with mean value 0, so is $\int_t^{\infty} e^{t-s}\varphi(s)\, ds + \int_{-\infty}^{t} e^{-(t-s)}\varphi(s)\, ds$.

Proof. Change (E1) to a system $(x' = y, y' = x + G(t))$ where $G(t) := g(t, x(t), x'(t-h), x''(t-h))$.

Express the system in vector notation as

$$X' = \begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} X + \begin{pmatrix} 0 \\ G(t) \end{pmatrix} =: AX + \Gamma(t).$$

Let

$$J = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

and transform $X' = AX + \Gamma(t)$ by $X = JY$ so that

$$Y' = J^{-1}AJY + J^{-1}\Gamma(t)$$

or

$$(y_1' = y_1 + \tfrac{1}{2}G(t), y_2' = -y_2 + \tfrac{1}{2}G(t)).$$

If $Y(t)$ is any bounded solution on $(-\infty, \infty)$, then

$$y_1(t) = -\frac{1}{2}\int_t^{\infty} e^{t-s} G(s)\, ds$$

and

$$y_2(t) = \frac{1}{2}\int_{-\infty}^{t} e^{-(t-s)} G(s)\, ds.$$

Taking $X = JY$ now gives (E2) and (E3) as the coordinates for X. Thus, any bounded solution of any of the three equations also satisfies the others. The last statement of the theorem requires only a simple calculation.

If g is independent of x' and x'', then (E2) is a standard integral equation which can define a fixed point mapping. We now fix (E3) up into the form of (1). Recall that

$$g(t,x,x',x'') = p(t) + r(t,x) + \beta x'(t-h) + \gamma x''(t-h).$$

Note that

$$\int_{-\infty}^{t} e^{-(t-s)}\gamma x''(s-h)\,ds =$$
$$\gamma x'(t-h) - \gamma\int_{-\infty}^{t} e^{-(t-s)}x'(s-h)\,ds.$$

The function $u(t,x'(\cdot))$ in (1) can now be identified as

$$\tfrac{1}{2}\Big[\int_{-\infty}^{t} e^{-(t-s)}(\beta-\gamma)x'(s-h)\,ds + 2\gamma x'(t-h)$$
$$-\int_{t}^{\infty} e^{t-s}(\beta+\gamma)x'(s-h)\,ds\Big].$$

Since $|\beta-\gamma| + 2|\gamma| + |\beta+\gamma| < 2$, $u(t,\varphi)$ defines a contraction mapping on P_T. If φ has mean value zero, so does $u(t,\varphi)$.

3. Periodic solutions. Let $T > 0$, $R := (-\infty,\infty)$, $(P_T, \|\cdot\|)$ be the Banach space of continuous T-periodic functions $\varphi : R \to R$ with the supremum norm, $\Delta^- := \{(t,s)|s \le t\}$, and $\Delta^+ := \{(t,s)|s \ge t\}$. Suppose that f, g, u, and p are continuous with $f : \Delta^- \times R \to R$, $g : \Delta^+ \times R \to R$, $p : R \to R$, and $u : R \times P_T \to R$.

Consider the equation

$$x'(t) = u(t,x'(\cdot)) + \int_{-\infty}^{t} f(t,s,x(s))\,ds + \int_{t}^{\infty} g(t,s,x(s))\,ds + p(t),\ t \in R. \tag{1}$$

where

$$p(t+T) = p(t), \int_0^T p(s)\,ds = 0,\ u(t+T,\varphi) = u(t,\varphi),$$
$$\|u(t,\varphi) - u(t,\psi)\| \le \alpha\|\varphi-\psi\| \text{ for } \alpha < 1,\ u(t,0) = 0, \tag{2}$$

and

$$f(t+T, s+T, x) = f(t, x) \text{ and } g(t+T, s+T, x) = g(t, x). \tag{3}$$

Moreover, for any $J > 0$ there are continuous functions $F_J : \Delta^- \to R^+ := [0, \infty)$ and $G_J : \Delta^+ \to R^+$ such that:

$$F_J(t+T, s+T) = F_J(t, s) \text{ if } (t, s) \in \Delta^-, \tag{3i}$$

$$|f(t, s, x)| \le F_J(t, s) \text{ if } (t, s) \in \Delta^- \text{ and } |x| \le J, \tag{3ii}$$

$$G_J(t+T, s+T) = G_J(t, s) \text{ if } (t, s) \in \Delta^+, \tag{3iii}$$

and

$$|g(t, s, x)| \le G_J(t, s) \text{ if } (t, s) \in \Delta^+ \text{ and } |x| \le J. \tag{3iv}$$

Also, assume that

$$\int_{-\infty}^{t-\tau} F_J(t, s)\, ds + \int_{t+\tau}^{\infty} G_J(t, s)\, ds \to 0 \tag{4}$$

uniformly for $t \in R$ as $\tau \to \infty$.

Recall that $(P_T, \|\cdot\|)$ is the Banach space of continuous T-periodic scalar functions with the supremum norm and we now let $(P_T^0, \|\cdot\|)$ be that subspace whose elements have mean value zero.

For any $J > 0$,

$$P_T(J) = \{\varphi \in P_T \mid \|\varphi\| \le J\}$$

and

$$P_T^0(J) = \{\varphi \in P_T^0 \mid \|\varphi\| \le J\}.$$

Next, define a map $A = A_k$ on P_T^0 by $\varphi \in P_T^0$ implies that

$$(A\varphi)(t) = \int_{-\infty}^{t} f(t,s,k+\int_0^s \varphi(u)\,du)\,ds + \int_t^{\infty} g(t,s,k+\int_0^s \varphi(u)\,du)\,ds + p(t),\ t \in R, \tag{5}$$

where k is a constant chosen so that $A\varphi \in P_T^0$. The following proposition gives conditions ensuring the existence of such a k.

Proposition 1. Let (2) – (4) hold, suppose that $\frac{\partial f}{\partial x}(t,s,x)$ and $\frac{\partial g}{\partial x}(t,s,x)$ are continuous, and that

$$f(t,s,0) \equiv 0 \text{ and } g(t,s,0) \equiv 0, \tag{6}$$

$$\frac{\partial f}{\partial x}(t,s,x)\frac{\partial g}{\partial x}(t,s,x) > 0 \text{ if } 0 < |x| \le JT \tag{7}$$

for some $J > 0$. Then for any $\varphi \in P_T^0(J)$ there is a unique k with $|k| \le \frac{T}{2}\|\varphi\|$ so that $A\varphi$ defined in (5) satisfies $A\varphi \in P_T^0$ and $|k + \int_0^t \varphi(s)\,ds| \le T\|\varphi\|$ for all t.

Proof. It is easy to see that $\varphi \in P_T^0$ implies that $|\int_0^t \varphi(s)\,ds| \le \frac{T}{2}\|\varphi\|$. (The continuous function $|\int_0^t \varphi(s)\,ds|$ has a maximum at $t_1 \in (0,T)$; thus, $|\int_0^{t_1} \varphi(s)\,ds| = |\int_{t_1}^{T} \varphi(s)\,ds|$ and either $t_1 \le T/2$ or $T - t_1 < T/2$.) This, together with (6) and (7), implies that

$$\Big\{ f(t,s,\frac{T\|\varphi\|}{2} + \int_0^s \varphi(u)\,du) \ge 0(\text{ or } \le 0) \text{ and } g(t,s,\frac{T\|\varphi\|}{2} + \int_0^s \varphi(u)\,du) \ge 0(\text{ or } \le 0)\Big\}$$

and

$$\Big\{ f(t,s,-\frac{T\|\varphi\|}{2} + \int_0^s \varphi(u)\,du) \le 0(\text{ or } \ge 0) \text{ and } g(t,s,-\frac{T\|\varphi\|}{2} + \int_0^s \varphi(u)\,du) \le 0(\text{ or } \ge 0)\Big\}$$

for all t and s. Now

$$\int_0^T \int_{-\infty}^{t} f(t,s,k+\int_0^s \varphi(u)\,du)\,ds\,dt + \int_0^T \int_t^{\infty} g(t,s,k+\int_0^s \varphi(u)\,du)\,ds\,dt + \int_0^T p(t)\,dt$$

is a continuous increasing (or decreasing) function of the constant k on $[-T\|\varphi\|/2, T\|\varphi\|/2]$. Thus, the required k with $|k| \leq T\|\varphi\|/2$ is uniquely assured, yielding $A\varphi \in P_T^0$ and $|k + \int_0^t \varphi(s)\,ds| \leq T\|\varphi\|$ for all t.

Now we have the following two propositions.

Proposition 2. *Let the conditions of Proposition 1 hold and for each $\varphi \in P_T^0(J)$ pick that unique k and define A by (5). Then A is continuous on $P_T^0(J)$.*

Proof. We will show that if $\varphi \in P_T^0(J)$ is fixed, if $\{\varphi_i\} \subset P_T^0(J)$, and if $\varphi_i \to \varphi$, then $A\varphi_i \to A\varphi$. By way of contradiction, if $A\varphi_i \not\to A\varphi$ then there is a subsequence, say φ_i again, and $\delta > 0$ with $\|A\varphi_i - A\varphi\| \geq \delta$. As $\varphi_i \to \varphi$, it is clear that $\int_0^t \varphi_i(s)\,ds \to \int_0^t \varphi s)\,ds$; thus, if k and k_i are the unique constants in the definition of $A\varphi$ and $A\varphi_i$, then $k_i \not\to k$. In particular, there is a subsequence, say k_i again, and a $\mu > 0$ with $|k_i - k| \geq \mu$. Thus, for each $s \in [0, T]$ there are $\xi(s)$ and $\eta(s)$ with

$$\begin{aligned}
0 &= \int_0^T \int_{-\infty}^t \left[f\left(t, s, k + \int_0^s \varphi(u)\,du\right) - f\left(t, s, k_i + \int_0^s \varphi_i(u)\,du\right)\right] ds\,dt \\
&+ \int_0^T \int_t^\infty \left[g\left(t, s, k + \int_0^s \varphi(u)du\right) - g\left(t, s, k_i + \int_0^s \varphi_i(u)\,du\right)\right] ds\,dt \\
&= \int_0^T \int_{-\infty}^t \frac{\partial f}{\partial x}(t, s, \xi(s)) \left[k - k_i + \int_0^s (\varphi(u) - \varphi_i(u))\,du\right] ds\,dt \\
&+ \int_0^T \int_t^\infty \frac{\partial g}{\partial x}(t, s, \eta(s)) \left[k - k_i + \int_0^s (\varphi(u) - \varphi_i(u))\,du\right] ds\,dt.
\end{aligned}$$

This is a contradiction since the right-hand-side is not zero when $|\int_0^t (\varphi(u) - \varphi_i(u))\,du| < \mu/2$ for all t.

Proposition 3. *Under the assumptions of Proposition 2, there is a continuous increasing positive function $\delta = \delta_J(\varepsilon) : (0, \infty) \to (0, \infty)$ with*

(8) $$|(A\varphi)(t_1) - (A\varphi)(t_2)| \leq \varepsilon \textit{ if } \varphi \in P_T^0(J) \textit{ and } t_1 < t_2 < t_1 + \delta.$$

Proof. First we prove that there is a continuous increasing positive function $\delta^- = \delta_J^-(\varepsilon) : (0, \infty) \to (0, \infty)$ with

(9) $$|(A^-\varphi)(t_1) - (A^-\varphi)(t_2)| \leq \varepsilon/3 \text{ if } \varphi \in P_T^0(J) \text{ and } t_1 < t_2 < t_1 + \delta^-,$$

where A^- is defined by

$$(A^-\varphi)(t) = \int_{-\infty}^{t} f\left(t, s, k + \int_0^s \varphi(u)du\right) ds, \ t \in R.$$

From (4), for any $\varepsilon > 0$ there is a $\tau > 0$ such that

$$\int_{-\infty}^{t-\tau} F_{JT}(t,s)\, ds \leq \varepsilon/12 \text{ if } t \in R. \tag{10}$$

For any $\varphi \in P_T^0(J)$, t_1 and t_2 with $t_1 < t_2$ we have

$$\begin{aligned}
&|(A^-\varphi)(t_1) - (A^-\varphi)(t_2)| \\
&=\left|\int_{-\infty}^{t_1} f\left(t_1, s, k + \int_0^s \varphi(u)\, du\right) ds - \int_{-\infty}^{t_2} f\left(t_2, s, k + \int_0^s (\varphi(u)\, du\right) ds\right| \\
&\leq \int_{t_1-\tau}^{t_1} \left| f\left(t_1, s, k + \int_0^s \varphi(u)\, du\right) - f\left(t_2, s, k + \int_0^s \varphi(u)\, du\right)\right| ds \\
&+ \int_{-\infty}^{t_1-\tau} F_{JT}(t_1,s)\, ds + \int_{-\infty}^{t_1-\tau} F_{JT}(t_2,s)\, ds + \int_{t_1}^{t_2} F_{JT}(t_2,s)\, ds \\
&\leq \int_{t_1-\tau}^{t_1} \left| f\left(t_1, s, k + \int_0^s \varphi(u)\, du\right) - f\left(t_2, s, k + \int_0^s \varphi(u)\, du\right)\right| ds \\
&+ \int_{t_1}^{t_2} F_{JT}(t_2,s)\, ds + \frac{\varepsilon}{6}.
\end{aligned} \tag{11}$$

Since $f(t,s,x)$ is uniformly continuous on $U = \{(t,s,x) | t-\tau-1 \leq s \leq t \text{ and } |x| \leq JT\}$, for the ε there is a δ_1 such that $0 < \delta_1 < 1$ and $|f(t_1,s,x) - f(t_2,s,x)| \leq \varepsilon/(12\tau)$ if $(t_1,s,x), (t_2,s,x) \in U$ and $|t_1 - t_2| < \delta_1$.

From this, if $t_1 < t_2 < t_1 + \delta_1$, then we obtain

$$\int_{t_1-\tau}^{t_1} \left| f\left(t_1, s, k + \int_0^s \varphi(u)\, du\right) - f\left(t_2, s, k + \int_0^s \varphi(u)\, du\right)\right| ds \leq \varepsilon/12. \tag{12}$$

Now let $F = \sup\{F_{JT}(t,s) | t-1 \leq s \leq t\}$. Then, for the ε there is a δ_2 such that $0 < \delta_2 < \min(\varepsilon/F, 1)$ and

$$\int_{t_1}^{t_2} F_{JT}(t_2,s)\, ds \leq \varepsilon/12 \text{ if } t_1 < t_2 < t_1 + \delta_2. \tag{13}$$

Thus, from (11) – (13), for the $\delta_3 = \min(\delta_1, \delta_2)$ we have (9) with $\delta^- = \delta_3$. Since we may assume that $\delta_3(\varepsilon)$ is nondecreasing, we can easily conclude that there is a continuous increasing function $\delta^- = \delta_J^- : (0, \infty) \to (0, \infty)$ which satisfies (9).

In the same way we can prove that there is a continuous increasing function $\delta^+ = \delta_J^+(\varepsilon) : (0, \infty) \to (0, \infty)$ with

$$|(A^+\varphi)(t_1) - (A^+\varphi)(t_2)| \le \varepsilon/3 \text{ if } \varphi \in P_T^0(J) \text{ and } t_1 < t_2 < t_1 + \delta^+, \tag{14}$$

where A^+ is defined by

$$(A^+\varphi)(t) = \int_t^\infty g\left(t, s, k + \int_0^s \varphi(u)\, du\right) ds,\ t \in R.$$

Finally, it is clear that there is a continuous increasing function $\delta_4 = \delta_4(\varepsilon) : (0, \infty) \to (0, \infty)$ with

$$|p(t_1) - p(t_2)| \le \varepsilon/3 \text{ if } t_1 < t_2 < t_1 + \delta_4. \tag{15}$$

It now follows from (9), (14), and (15) that (8) holds for $\delta = \min(\delta^-, \delta^+, \delta_4)$.

Here is our main result.

Theorem 2. *Let the conditions of Proposition 1 hold. Suppose there is an $\alpha < 1$ such that*

$$\begin{cases} \varphi, \phi \in P_T^0 \text{ imply that } \|u(t, \varphi) - u(t, \phi)\| \le \alpha \|\varphi - \phi\|, \\ u(t, \varphi) \in P_T^0, u(t, 0) = 0, \end{cases} \tag{16}$$

$$\alpha J + \int_{-\infty}^t F_{JT}(t, s)\, ds + \int_t^\infty G_{JT}(t, s)\, ds + \|p\| \le J. \tag{17}$$

Then (1) has a T-periodic solution.

Proof. Let $S = P_T^0(J)$ and define a map B on S by $\varphi \in S$ implies that $(B\varphi)(t) = u(t, \varphi(\cdot))$.

Then S is a closed convex non-empty subset of a Banach space P_T^0. Proposition 1 and (17) imply that A and B are well-defined and map S into S, while φ and ψ in S yield $A\varphi + B\psi \in S$. By Proposition 2 the function A is continuous. Proposition 3 shows that the set $\{A\varphi | \varphi \in S\}$ is equicontinuous. Since B is a contraction the conditions of Krasnoselskii's result (cf. Smart [3; p. 31]) are satisfied and there exists $\varphi \in S$ with $A\varphi + B\varphi = \varphi$. Thus, $\Phi = k + \int_0^t \varphi(s)\, ds \in P_T$ is a T-periodic solution of (1) and the proof is complete.

EQUATION (E1) revisited. We focus again on

$$(E3) \qquad x'(t) = \frac{1}{2}\Big[\int_{-\infty}^{t} e^{-(t-s)} g(s, x(s), x'(s-h), x''(s-h))\, ds \\ - \int_t^{\infty} e^{t-s} g(s, x(s), x'(s-h), x''(s-h))\, ds\Big]$$

and

$$g(t, x, x', x'') = p(t) + r(t, x) + \beta x'(t-h) + \gamma x''(t-h).$$

Take $T = 2\pi$, $p(t + 2\pi) = p(t)$,

$$(E4) \qquad |\beta - \gamma| + 2|\gamma| + |\beta + \gamma| < 2$$

and

$$(E5) \qquad r(t, x) = \text{ Arctan } x.$$

The conditions of the Theorem 2 are readily satisfied since the function A maps $P_{2\pi}^0$ into a bounded set. Thus, (E1), (E2), and (E3) all have a 2π-periodic solution.

We turn now to two more direct examples of the theorem, one linear and one nonlinear.

EXAMPLE 1. Consider the scalar linear equation

$$(18) \qquad x'(t) = \alpha x'(t-h) + \beta \int_{-\infty}^{t} e^{s-t}(2 + \cos t) x(s)\, ds \\ + \gamma \int_t^{\infty} e^{t-s}(2 + \sin t) x(s)\, ds + p(t), \ t \in R,$$

where α, β, and γ are constants such that $\beta\gamma > 0$ and $|\alpha| + 6\pi|\beta + \gamma| < 1$, and $p : R \to R$ is a continuous 2π-periodic function with $\int_0^{2\pi} p(s)\,ds = 0$. Equation (18) is obtained from (1) taking $T = 2\pi$, $f(t, s, x) = \beta x(2 + \cos t)e^{s-t}$ and $g(t, s, x) = \gamma x(2 + \sin t)e^{t-s}$. Let J be a number satisfying $\|p\| \le (1 - |\alpha| - 6\pi|\beta + \gamma|)J$. For this J we can take the following functions as F_J and G_J:

$$F_J(t, s) = 3|\beta|Je^{s-t},\ (t, s) \in \Delta^-$$

and

$$G_J(t, s) = 3|\gamma|Je^{t-s},\ (t, s) \in \Delta^+.$$

It is easy to see that (2) – (4), (6), (7), and (17) with $T = 2\pi$ are satisfied. Thus, by Theorem 2, (18) has a 2π-periodic solution in $P_{2\pi}(2\pi J)$.

EXAMPLE 2. Corresponding to (18), consider the scalar nonlinear equation

$$x'(t) = \alpha x'(t - h) + \beta \int_{-\infty}^{t} e^{s-t}(2 + \cos t)x^3(s)\,ds + \gamma \int_t^{\infty} e^{t-s}(2 + \sin t)x^3(s)\,ds + p(t),\ t \in R, \tag{19}$$

where α, β, and γ are constants and $p : R \to R$ is a continuous 2π-periodic function. Let $\beta\gamma > 0$ and $|\alpha|J + 24\pi^3|\beta + \gamma|J^3 + \|p\| \le J$ for some $J > 0$. Equation (19) is obtained from (1) by taking $T = 2\pi$, $f(t, s, x) = \beta x^3(2 + \cos t)e^{s-t}$ and $g(t, s, x) = \gamma x^3(2 + \sin t)e^{t-s}$. For the J we can take the following functions as F_J and G_J:

$$F_J(t, s) = 3|\beta|J^3e^{s-t},\ (t, s) \in \Delta^-$$

and

$$G_J(t, s) = 3|\gamma|J^3e^{t-s},\ (t, s) \in \Delta^+.$$

Clearly, (2) – (4), (6), (7), and (17) are satisfied when $T = 2\pi$. By Theorem 2, (19) has a 2π-periodic solution.

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