BASIC NEUTRAL INTEGRAL EQUATIONS OF ADVANCED TYPE

T.A. Burton
Department of Mathematics
Southern Illinois University
Carbondale, IL 62901-4408

ABSTRACT. In this paper we study problems of the form

$$x(t) = f(x(t-h)) + \int_t^\infty Q(s, x(s), x(s-h))C(t,s)ds + p(t)$$

with a view to showing the existence of a solution in a given space. The space is dictated by the magnitude and properties of p and the convergence properties of C. We focus on periodic solutions and solutions depending on an initial function. The latter are usually discontinuous, but the jumps tend to zero.

1. Introduction. Neutral differential equations have been studied with varying degrees of vigor since the middle of the century (cf. [3–6], [8–10]). Recently, several investigators have given heuristic arguments to support their use in describing biological phenomena (cf. [5–6], [8–10]) and much of this is formalized in the final chapter of each of the books by Gopalsamy [5] and Kuang [8].

In such studies a specific type of solution is sought, such as a periodic solution as the attractor for a logistic equation. And effective search for particular types of solutions most frequently is accomplished by converting the differential equation to an integral equation and then applying fixed point theory. From such considerations we are led to consider neutral integral equations of types about which almost nothing seems to be known. We establish basic theory of existence, periodicity, and properties of solutions by means of fixed point theory.

The forms of the problems we consider are motivated in two ways. First, we look at neutral functional differential equations which are currently generating much investigation and write them as integral equations. This establishes the basic form to be studied, although the motivating equations themselves are local in nature. With those forms in mind there is a clear need to simplify them into equations which are nearly linear, or at most polynomial, and about which general theory can be formulated. This is the scope of the present paper. In the last section we briefly describe how the original nonlinear problem can be attacked.

Throughout, if a function is written without its argument, that argument is t.

2. Motivation and summary. Consider the "logistic equation"

$$x' = ax + \alpha x'(t-h) - q(t, x, x(t-h)) \tag{1}$$

where $a > 0$, $0 \leq |\alpha| < 1$, $h > 0$, all are constant. We are interested in a solution for $t \geq 0$, possibly arising from a given initial function, or for a solution on $(-\infty, \infty)$, possibly a periodic solution. Write (1) as

$$(x - \alpha x(t-h))' = a(x - \alpha x(t-h)) + a\alpha x(t-h) - q(t, x, x(t-h)),$$

multiply by e^{-at}, and group terms as

$$\big[(x - \alpha x(t-h))e^{-at}\big]' = \big[a\alpha x(t-h) - q(t, x, x(t-h))\big]e^{-at}.$$

We search for a solution having the property that

$$(x(t) - \alpha x(t-h))e^{-at} \to 0 \text{ as } t \to \infty$$

so that an integration from t to infinity yields

$$-(x(t) - \alpha x(t-h))e^{-at} = \int_t^\infty \big[a\alpha x(s-h) - q(s, x(s), x(s-h))\big]e^{-as} ds$$

and, finally

$$\text{(1I)} \qquad x(t) = \alpha x(t-h) + \int_t^\infty \big[q(s, x(s), x(s-h)) - a\alpha x(s-h)\big] e^{a(t-s)} ds.$$

This form motivates our study in Section 3 and we focus on an equation

$$\text{(2)} \qquad x(t) = f(x(t-h)) + \int_t^\infty Q(s, x(s), x(s-h)) C(t-s) ds + p(t)$$

(sometimes $C = C(t,s)$) in which

$$\text{(3)} \qquad |f(x) - f(y)| \le \alpha |x-y|, \quad 0 \le \alpha < 1,$$

and for some fixed k with $0 \le k \le 1$

$$\text{(4)} \qquad |Q(t,x,y) - Q(t,w,z)| \le (k|x-w| + (1-k)|y-z|),$$

$$\text{(5)} \qquad |Q(t,0,0)| \le 1,$$

$$\text{(6)} \qquad Q,\ p,\ \text{and } C \text{ are continuous}, \ \int_0^\infty |C(-u)| du =: C_0 < \infty.$$

Equation (2) can be called a neutral delay integral equation of advanced type.

Two remarks about the form of (2) are of interest. In the study of a paper of Schauder, Krasnoselskii is said to have observed (cf. [12; p. 31]) that the inversion of a perturbed differential operator yields the sum of a contraction operator and a compact operator. That is precisely the case here and a fixed point theorem of Krasnoselskii type plays a main role in this paper.

The second remark concerns the integral from t to infinity; while not common, it can be found in many places. Coddington and Levinson [2; p. 331] write an ordinary differential equation in this way (without h) when studying an unstable manifold. Also, investigators have long written differential equations

$$x' = F(t,x)$$

as

$$x(t) = x(t_0) + \int_{t_0}^t F(s, x(s)) ds.$$

Then, if it can be determined that every solution converges to zero as $t \to \infty$, it is permissible to bypass $x(t_0)$ and write

$$x(t) = -\int_t^{\infty} F(s, x(s))ds.$$

While we see many equations of the form of (2) (without h), such equations usually are studied only after existence of solutions has been established. That is a central question here.

Differential equations usually satisfy existence theorems on a short interval, say $[t_0, L]$; then the solution may be extended to $[t_0, \infty)$ in a variety of ways, but mainly by showing that the solution remains in a given region, so long as it exists. Lakshmikantham and Leela [11; p. 46] have shown immediately global existence, but that is unusual and is an option. Here, it is not an option. We must immediately establish existence on $[t_0, \infty)$.

And because of this, the central problem for (2) is to determine the space in which solutions reside. Once that is done, the actual proof of existence follows from application of a fixed point theorem. Moreover, unlike the case just mentioned of a differential equation, the existence of a solution of (2) in a given space endows that solution with the properties of that space. Thus, proof of existence yields important qualitative properties as well. We illustrate that in the examples.

Equation (2) holds many surprises. For functional differential equations of neutral type, it may be deduced from Driver [3] that repeated discontinuities of a solution on $[t_0, \infty)$ must be expected as a result of a given initial function. We note that this is true for (2); but we also show that the magnitude of the jumps in the discontinuities tends to zero as $t \to \infty$. This is parallel to an interesting phenomena for delay-differential equations. El'sgol'ts [4; p. 7] discusses solutions of

$$x' = H(t, x(t), x(t-h))$$

in which H is in C^{∞} and there is an initial function $\overline{\phi} : [-h, 0] \to R$ yielding a solution $x(t, 0, \overline{\phi})$ on $[0, \infty)$. He notes that $x^{(k)}(t, 0, \overline{\phi})$ may have a discontinuity at $(k-1)h$, but will be continuous for $t > kh$; the solution smooths with time.

Investigators study integral equations of the form

$$x(t) = a(t) + \int_0^t Q(t, s, x(s))ds.$$

The frequent goal is to show that for well-behaved kernels, the solution follows $a(t)$ in some sense. This is an off-shoot of the elementary method of undetermined coefficients for differential equations with constant coefficients. Let $x_1(t)$ and $x_2(t)$ be two linearly independent solutions of

$$x'' + x' + x = 0$$

and let $X(t)$ be any solution of

$$x'' + x' + x = p(t).$$

Then every solution of the forced equation can be written as

$$x(t) = c_1x_1(t) + c_2x_2(t) + X(t) \to X(t)$$

as $t \to \infty$. If p is polynomial, X is polynomial. If p is periodic, so is X. If p is exponential, then X is of exponential order.

This leads us to basic information about (2). As examples:

(a) If p is bounded, we construct a Banach space of bounded functions and find a solution of (2) in that space; so it is bounded. In particular, periodic solutions are found.

(b) If p grows of order of a polynomial, say $p(t) = t$, we construct a Banach space with weighted norm, say $|\phi| := \sup_{t\geq -h} |\phi(t)|/(t + 1 + h)$, and find a solution in that space.

The magnitude of p induces a norm. If that norm is compatible with the magnitude of the kernel in (2), then a fixed point of a map results which is a solution of (2).

In Section 4 we leave the small kernels and focus on equations with large kernels and with $p(t)$ periodic. Using a generalization of a fixed point theorem which combines those of Krasnoselskii and Schaefer we prove the existence of periodic solutions. This involves obtaining a priori bounds by means of Liapunov functionals.

This paper represents the merest beginnings of a theory of neutral integral equations. Had we begun with a neutral predator-prey system

$$x' = \alpha x'(t-h) + ax - q(t, x, y, x(t-h), y(t-h))$$
$$y' = \beta y'(t-h) - ky(t) + r(t, x, y, x(t-h), y(t-h))$$

then we would have arrived at a system

$$X(t) = F(t, X(t-h), \int_{-\infty}^{t} G(s, X(s), X(s-h))ds,$$
$$\int_{t}^{\infty} H(s, X(s), X(s-h))ds).$$

This introduces a neutral integral equation of retarded and advanced type. Even when h is zero, such equations represent common classes of equations of the utmost importance. They clearly deserve study.

3. Small kernels. Consider again

$$x(t) = f(x(t-h)) + \int_{t}^{\infty} Q(s, x(s), x(s-h))C(t-s)ds + p(t) \tag{2}$$

with (3)–(6) holding.

THEOREM 1. Let p be bounded on R, let (3)–(6) hold, and suppose that

$$\mu := \alpha + C_0 < 1, \tag{7}$$

where α and C_0 are in (3) and (6). Then (2) has a unique bounded continuous solution on $-\infty < t < \infty$.

PROOF. Let $(\mathcal{B}, \|\cdot\|)$ be the Banach space of bounded continuous functions $\phi : R \to R$ with the supremum norm. Define $P : \mathcal{B} \to \mathcal{B}$ by $\phi \in \mathcal{B}$ implies that

$$(P\phi)(t) = f(\phi(t-h)) + \int_{t}^{\infty} Q(s, \phi(s), \phi(s-h))C(t-s)ds + p(t). \tag{8}$$

As ϕ is bounded and continuous, so is $Q(t,\phi(t),\phi(t-h))$ using (4)–(6); since C is $L^1[0,\infty)$ it follows from a theorem of Hewitt and Stromberg [7; p. 398] that the integral is uniformly continuous. That integral is bounded by (6). Hence, $P\phi$ is bounded and continuous.

Next, if $\phi,\psi\in\mathcal{B}$ then

$$\begin{aligned}
&|(P\phi)(t)-(P\psi)(t)|\\
&\quad\le\alpha\|\phi-\psi\|+\int_t^\infty[k|\phi(s)-\psi(s)|+(1-k)|\phi(s-h)-\psi(s-h)|]|C(t-s)|ds\\
&\quad\le\alpha\|\phi-\psi\|+\|\phi-\psi\|\int_t^\infty|C(t-s)|ds\\
&\quad=\mu\|\phi-\psi\|.
\end{aligned}$$

As $\mu<1$, P is a contraction with unique fixed point. This completes the proof.

COR. If the conditions of Theorem 1 hold and if there is a $T>0$ such that $Q(t,x,y)=Q(t+T,x,y)$ and $p(t+T)=p(t)$, then (2) has a unique T-periodic solution.

PROOF. A change of variable shows that if $(\mathcal{B},\|\cdot\|)$ is the Banach space of continuous T-periodic functions with the supremum norm, always denoted by

$$(\mathcal{P}_T,\|\cdot\|),$$

then $\phi\in\mathcal{P}_T$ implies that $P\phi\in\mathcal{P}_T$. Thus, the fixed point is in $\mathcal{P}_T$.

It is natural to ask if, under the conditions of Theorem 1, (2) might also have an unbounded solution, $D(t)$, on $(-\infty,\infty)$. A study of the work up to Theorem 2 will show this is possible only if $\int_t^\infty|C(t-s)||D(s)|ds$ is large. We continue this question after Example 1.

The solutions just obtained are continuous and satisfy (2) on R. A different point of view studies (2) and an initial condition. Suppose that there is a given initial function $\overline{\phi}:[-h,0]\to R$ which is continuous. The initial value problem for (2) asks that we find a function $x:[-h,\infty)\to R$, denoted by $x(t,0,\overline{\phi})$, with $x(t,0,\overline{\phi})=\overline{\phi}(t)$ on $[-h,0)$ and x satisfies (2) on $[0,\infty)$.

REMARK 1. If $x(t) = x(t, 0, \overline{\phi})$ is to be continuous at $t = 0$, then we must have

$$x(0) = \overline{\phi}(0) = f(\overline{\phi}(-h)) + \int_0^\infty Q(s, x(s), x(s-h))C(-s)ds + p(0). \tag{9}$$

Thus, for an arbitrary continuous $\overline{\phi}$, a finite jump discontinuity at $t = 0$ must be expected. Under conditions to be given, the integral in (2) will be continuous whenever x is piecewise continuous and so, from (2), we must expect x to have jump discontinuities at nh, for $n = 1, 2, \ldots$ This shows us how to search for a solution on $[0, \infty)$.

First, examine $p(t)$ and select a constant $K > 0$ and a continuous function $D : [-h, \infty) \to [1, \infty)$ with

$$\sup_{t \geq -h} |p(t)/D(t)| < \infty \text{ and } D(t-h)/D(t) \leq K. \tag{10}$$

The function D will be the weight for a norm on a Banach space and notation is simplified if D is increasing, but we do not ask that. We will see that we want $D(t)$ to be as small as possible.

Next, for D to be compatible with $C(t-s)$ we will need

$$\sup_{t \geq 0} \int_t^\infty |C(t-s)|[D(s)/D(t)]ds < \infty. \tag{11}$$

If we can not satisfy (10) and (11), then we are unable to say anything about (2) in this section.

Finally, we will want a mapping induced by (2) to map piecewise continuous functions into piecewise continuous functions. This will lead us to ask that

$$\int_t^\infty Q(s, \phi(s), \phi(s-h))C(t-s)ds$$

be continuous whenever ϕ is piecewise continuous and $\phi(t)/D(t)$ is bounded. We have already noted in the proof of Theorem 1 that if ϕ is bounded and piecewise continuous then a classical theorem says that this integral is uniformly continuous. We ask that

$$\begin{cases} \forall J > 0, \text{ if } 0 \leq t_1 < t_2 \leq J, \text{ then} \\ \displaystyle\int_{t_2}^\infty |C(t_1 - s) - C(t_2 - s)|D(s)ds \to 0 \text{ as } |t_1 - t_2| \to 0. \end{cases} \tag{12}$$

In order to have a contraction mapping we also need

$$\alpha K + (k + (1-k)K) \sup_{t\geq 0} \int_t^\infty |C(t-s)|[D(s)/D(t)]ds =: \mu < 1. \tag{13}$$

Define

$$(\mathcal{B}, |\cdot|_D) \tag{14}$$

as the Banach space of functions $\phi : [-h, \infty) \to R$ which are continuous on $[(n-1)h, nh)$ with left-hand limits existing at nh and with the property that

$$|\phi|_D := \sup_{t\geq -h} |\phi(t)/D(t)| \text{ exists.} \tag{15}$$

THEOREM 2. Let (10)–(13) hold and let $\overline{\phi} : [-h, 0] \to R$ be a given continuous function. Then there is a unique $x \in \mathcal{B}$ satisfying (2) for $t \geq 0$ and $x(t) = \overline{\phi}(t)$ on $[-h, 0)$.

PROOF. Let

$$\mathcal{B}^* = \{\phi \in \mathcal{B} \,|\, \phi(t) = \overline{\phi}(t) \text{ on } [-h, 0)\}.$$

Then $(\mathcal{B}^*, |\cdot|_D)$ is a complete metric space. Define $P : \mathcal{B}^* \to \mathcal{B}^*$ by $\phi \in \mathcal{B}^*$ implies that

$$\begin{cases} (P\phi)(t) = \overline{\phi}(t) \text{ if } -h \leq t < 0, \\ (P\phi)(t) = f(\phi(t-h)) + \displaystyle\int_t^\infty Q(s, \phi(s), \phi(s-h))C(t-s)ds + p(t), \quad t \geq 0. \end{cases} \tag{16}$$

Notice that

$$|f(x)| \leq |f(0)| + \alpha|x|$$

and

$$\begin{aligned} |Q(t,x,y)| &\leq |Q(t,0,0)| + k|x| + (1-k)|y| \\ &\leq 1 + k|x| + (1-k)|y| \end{aligned}$$

by (3)–(5) and so using (10) and (11) we have

$$\begin{aligned} |(P\phi)(t)/D(t)| \leq \big[|f(0)|/D(t)\big] &+ \alpha K[|\phi(t-h)|/D(t-h)] \\ &+ \int_t^\infty \big[(1/D(s)) + k(|\phi(s)|/D(s)) \\ &+ (1-k)K(|\phi(s-h)|/D(s-h))\big]|C(t-s)|(D(s)/D(t))ds \\ &+ |p(t)|/D(t). \end{aligned}$$

Taking the supremum for $t \geq 0$ we see that $|P\phi|_D$ exists.

To see that the integral in (16) is continuous, if $J > 0$ and if $0 \leq t_1 < t_2 \leq J$, then

$$\begin{aligned}
&\left| \int_{t_1}^{\infty} Q(s, \phi(s), \phi(s-h)) C(t_1 - s) ds - \int_{t_2}^{\infty} Q(s, \phi(s), \phi(s-h)) C(t_2 - s) ds \right| \\
&\quad \leq \left| \int_{t_1}^{t_2} Q(s, \phi(s), \phi(s-h)) C(t_1 - s) ds \right| \\
&\qquad + \int_{t_2}^{\infty} |Q(s, \phi(s), \phi(s-h))| |C(t_1 - s) - C(t_2 - s)| ds.
\end{aligned}$$

The first term on the R-H-S can be made small if $|t_1 - t_2|$ is small. The last term is bounded by

$$\int_{t_2}^{\infty} \big[|Q(s, \phi(s), \phi(s-h))|/D(s)\big] |C(t_1 - s) - C(t_2 - s)| D(s) ds.$$

The first factor in the integrand is bounded. By (12) the integral tends to zero as $|t_1 - t_2| \to 0$. Hence, the integral is continuous and $P\phi \in \mathcal{B}^*$.

Finally, we show that P is a contraction. If $\phi, \psi \in \mathcal{B}^*$, then

$$\begin{aligned}
&|(P\phi)(t) - (P\psi)(t)|/D(t) \\
&\quad \leq \alpha K |\phi - \psi|_D + \int_t^{\infty} \big\{ k \big[|\phi(s) - \psi(s)|/D(s) \big] \\
&\qquad + (1-k) K [|\phi(s-h) - \psi(s-h)|/D(s-h)] \big\} |C(t-s)| [D(s)/D(t)] ds \\
&\quad \leq \alpha K |\phi - \psi|_D + (k|\phi - \psi|_D + (1-k) K |\phi - \psi|_D) \int_t^{\infty} |C(t-s)| [D(s)/D(t)] ds \\
&\quad \leq \mu |\phi - \psi|_D,
\end{aligned}$$

as required. Hence, P is a contraction with unique fixed point $\phi \in \mathcal{B}^*$ and the proof is complete.

The first candidates which come to mind for $C(t)$ are βe^{at} and $\beta/(1 + t^{2n})$, $n \geq 1$. We illustrate the results with the first one. Consider the equation

$$x(t) = \alpha x(t-h) + \beta \int_t^{\infty} x(s) e^{a(t-s)} ds + p(t). \tag{17}$$

Clearly, if p is differentiable, it could be reduced to a neutral functional differential equation. We now give three forms for $p(t)$.

EXAMPLE 1. If $p(t)$ is bounded and continuous on R, then (17) has a unique bounded continuous solution on R provided that

$$(7^*) \qquad \mu = \alpha + |\beta| \int_t^\infty e^{a(t-s)} ds = \alpha + (|\beta|/a) < 1.$$

Question: Is it possible that with $p(t)$ bounded, then (17) can have an unbounded solution? By following the next examples we can see that the solution of Example 1 may be the only solution of (17) which is bounded by $Me^{\theta at}$, $0 < \theta < 1$, for $t > 0$ and some $M > 0$. If there are larger solutions, questions of convergence will arise.

EXAMPLE 2. If $p(t) = t$, then $D(t) = t + 1 + h$, $D(t-h)/D(t) \leq 1 =: K$, and

$$(7^{**}) \qquad \mu = \alpha K + |\beta| \sup_{t \geq 0} [at + 1 + a + ah]/a^2(t + 1 + h).$$

If, for example, $a = 1$, then $\mu < 1$ if and only if $\alpha + |\beta| < 1$. When $\mu < 1$, then for any continuous initial function $\overline{\phi} : [-h, 0] \to R$, there is a unique solution $x(t) = x(t, 0, \overline{\phi})$ of (17) on $[0, \infty)$ satisfying $\sup_{t \geq 0} |x(t)/(t + 1 + h)| < \infty$.

PROOF. Clearly, (11) will be satisfied. To verify (12) we have

$$\begin{aligned} &\int_{t_2}^\infty |e^{a(t_1 - s)} - e^{a(t_2 - s)}|(s + 1 + h) ds \\ &\qquad = \int_{t_2}^\infty e^{-as} |e^{at_1} - e^{at_2}|(s + 1 + h) ds \end{aligned}$$

so that if $J > 0$ is given and if $0 \leq t_1 < t_2 \leq J$ then this quantity tends to zero as $|t_1 - t_2| \to 0$. For (13) we examine

$$\int_t^\infty e^{-as}[s + 1 + h] ds = e^{-at}[at + 1 + a + ah]/a^2$$

so that

$$\begin{aligned} \int_t^\infty |C(t-s)|[D(s)/D(t)] ds &= |\beta|[e^{at}/(t + 1 + h)]e^{-at}[at + 1 + a + ah]/a^2 \\ &= |\beta|[at + 1 + a + ah]/a^2(t + 1 + h), \end{aligned}$$

as required.

EXAMPLE 3. Let $0 < c < a$, $p(t) = e^{ct}$ and $D(t) = e^{ct}e^{ch}$. Then $D(t-h)/D(t) = e^{ct}/e^{ct}e^{ch} = e^{-ch} = K$ and

$$\mu = \alpha e^{-ch} + (|\beta|/(a-c)). \tag{7***}$$

If $\mu < 1$, then for any given continuous $\overline{\phi} : [-h, 0] \to R$, there is a unique solution $x(t) = x(t, 0, \overline{\phi})$ of (17) satisfying $\sup_{t\geq 0} |x(t)/e^{ct}| < \infty$. The proof is a routine calculation.

REMARK 2. Theorem 1 and Example 1 can be stated far more strongly. When (3)–(6) hold, if (2) has a bounded solution then the integral in (2) is bounded for $t \geq 0$ and $x(t) - f(x(t-h))$ is bounded for $t \geq 0$. Hence, if we suppose that (3)–(7) hold, then (2) has a solution bounded for $t \geq 0$ if and only if $p(t)$ is bounded. Parallel remarks hold for the other results.

REMARK 3. There is an interesting parallel between discontinuities of (2) at nh and the discontinuities in the derivatives of solutions of a delay-differential equation

$$x' = f(t, x(t), x(t-h)) \tag{*}$$

where $f \in C^{\infty}$. According to El'sgol'ts [4; p. 7], for a given initial function $\overline{\phi} : [-h, 0] \to R$, if the solution $x(t) = x(t, 0, \overline{\phi})$ exists on $[0, \infty)$, then $x^{(k)}(t)$ may have a discontinuity at $(k-1)h$, but it is continuous for $t > (k-1)h$; that is, x smooths with increasing time. We have reasoned that the solution of (2), $x(t, 0, \overline{\phi})$, may be discontinuous at nh. We now see that the jumps become smaller as t becomes large.

THEOREM 3. Let $\mathcal{B}^*$ be defined in the proof of Theorem 2 and let (12) hold. If $0 \leq \alpha < 1$ and $x \in \mathcal{B}^*$ satisfies (2) for $t \geq 0$, then $x(nh^-) - x(nh^+) \to 0$ as $n \to \infty$.

PROOF. In the proof of Theorem 2 we noted that the integral in (2) is continuous. Since p is continuous, so is $x(t) - f(x(t-h))$. Thus,

$$x(nh^-) - f(x((n-1)h^-) = x(nh^+) - f(x((n-1)h^+))$$

or

$$\begin{aligned}
|x(nh^-) - x(nh^+)| &= |f(x((n-1)h^-) - f(x((n-1)h^+)| \\
&\le \alpha|x((n-1)h^-) - x((n-1)h^+)| \\
&\le \alpha|f(x((n-2)h^-) - f(x((n-2)h^+))| \\
&\le \alpha^2|x((n-2)h^-) - x((n-2)h^+)| \\
&\le \cdots \\
&\le \alpha^{n-1}|x(h^-) - x(h^+)| \\
&\to 0 \text{ as } n \to \infty,
\end{aligned}$$

completing the proof.

4. Large kernels: a Krasnoselskii-Schaefer type theorem. We now consider an extension of (2) in which there is less restriction on the size of the kernel, but far more on the form.

Let

$$x(t) = f(x(t-h)) + \int_t^\infty \big[g(x(s)) + r(x(s-h))\big]C(t,s)ds + p(t) \tag{18}$$

in which

$$|f(x) - f(y)| \le \alpha|x-y|,\ 0 \le \alpha < 1,\ xg(x) > 0 \text{ if } x \ne 0, \tag{19}$$

$$\begin{gathered} g, r, C, C_s, C_{st}, p \text{ are continuous, } p(t+T) = p(t), \\ C(t+T, s+T) = C(t,s), \end{gathered} \tag{20}$$

and

$$\begin{gathered} \sup_{t\ge 0}\left[\int_t^\infty |C(t,s)|ds + \int_t^\infty (|C_s(t,s)| + |C_{st}(t,s)|)(t-s)^2 ds\right] < \infty, \\ |C(t,s)||t-s| \to 0 \text{ as } s \to \infty. \end{gathered} \tag{21}$$

In addition to (18) we consider the homotopy equation

$$x(t) = \lambda\left[f(x(t-h)/\lambda) + \int_t^\infty \big[g(x(s)) + r(x(s-h))\big]C(t,s)ds + p(t)\right], \quad 0 < \lambda \le 1. \tag{22}$$

We will obtain an a priori bound on all T-periodic solutions of (22) for $0 < \lambda \leq 1$ and then use an extension of Schaefer's theorem in the direction of Krasnoselskii's theorem (cf. Smart [12; pp. 29–31]) to prove that (22) has a T-periodic solution.

LEMMA 1. If $x(t)$ is a continuous T-periodic solution of (22) and if V is defined by

$$V(t) = \int_t^\infty \lambda^2 C_s(t,s)\Big(\int_s^t \big[g(x(u)) + r(x(u-h))\big]du\Big)^2 ds, \tag{23}$$

then

$$\begin{aligned} V'(t) = \lambda^2 &\int_t^\infty C_{st}(t,s)\Big(\int_s^t \big[g(x(u)) + r(x(u-h))\big]du\Big)^2 ds \\ &+ 2\lambda\big(g(x) + r(x(t-h))\big)\big(x(t) - \lambda f(x(t-h)/\lambda) - \lambda p(t)\big). \end{aligned} \tag{24}$$

PROOF. The first term is clear. In addition we have

$$\begin{aligned} &2\lambda^2\big[g(x) + r(x(t-h))\big]\int_t^\infty C_s(t,s)\int_s^t \big[g(x(u)) + r(x(u-h))\big]du\,ds \\ &\quad = 2\lambda^2\big[g(x) + r(x(t-h))\big]\Big\{C(t,s)\int_s^t \big[g(x(u)) + r(x(u-h))\big]du\Big|_t^\infty \\ &\qquad + \int_t^\infty C(t,s)\big[g(x(s)) + r(x(s-h))\big]ds\Big\} \\ &\quad = 2\lambda\big[g(x) + r(x(t-h))\big][x - \lambda f(x(t-h)/\lambda) - \lambda p(t)], \end{aligned}$$

as required.

REMARK 4. The reader who is familiar with Liapunov's direct method needs to be alerted to the fact that we use V' alone (not V) to obtain a priori bounds on solutions of (22). The goal is to show that a modification of V, say W, satisfies

$$W'(t) \leq \lambda(-K|xg(x)| + M), \text{ some } K > 0,\ M > 0,$$

or

$$W'(t) \geq \lambda(K|xg(x)| - M).$$

As $x \in \mathcal{P}_T$ implies that $W \in \mathcal{P}_T$ we obtain $W(T) = W(0)$ and so an integration of *either* inequality yields

$$\int_0^T |x(t)g(x(t))|dt \leq M/K.$$

This inequality is then parlayed into an a priori bound on the supremum of x using (22), (21), and a Schwarz inequality. A fixed point theorem then yields a T-periodic solution of (18).

We now specialize further and write (22) as

$$x(t) = \alpha x(t-h) + \lambda\left[\int_t^\infty (\beta x^{2n+1}(s) + \gamma x^m(s-h))C(t,s)ds + p(t)\right] \tag{25}$$

so that (24) becomes

$$\begin{aligned} V'(t) = \lambda^2 \int_t^\infty C_{st}(t,s)\left(\int_s^t [\beta x^{2n+1}(u) + \gamma x^m(u-h)]du\right)^2 ds \\ + 2\lambda(\beta x^{2n+1} + \gamma x^m(t-h))(x - \alpha x(t-h) - \lambda p(t)). \end{aligned} \tag{26}$$

We will need to work with the last term which we write as

$$\begin{aligned} Q := 2\lambda[\beta x^{2n+2} - \alpha\beta x^{2n+1}x(t-h) + \gamma x x^m(t-h) \\ - \gamma\alpha x^{m+1}(t-h) - \lambda\beta x^{2n+1}p - \gamma\lambda x^m(t-h)p]. \end{aligned} \tag{27}$$

LEMMA 2. If $C_{st} \le 0$, if $\beta < 0$, and if $2n > m$, then there is a constant X so that if x is a T-periodic solution of (25), $0 < \lambda \le 1$, then

$$\int_0^T x^{2n+2}(s)ds \le X. \tag{28}$$

PROOF. We need two basic inequalities for Q:

$$|x^{2n+1}x(t-h)| \le [(2n+1)x^{2n+2} + x^{2n+2}(t-h)]/(2n+2)$$

and

$$|x^m(t-h)x| \le [m|x(t-h)|^{m+1} + |x|^{m+1}]/(m+1).$$

With these, from (27) we get

$$\begin{aligned} Q \le 2\lambda\Big[&\beta x^{2n+2} + \frac{|\alpha\beta|}{2n+2}(2n+1)x^{2n+2} \\ &+ \frac{|\alpha\beta|}{2n+2}x^{2n+2}(t-h) + \frac{|\gamma|m}{m+1}|x(t-h)|^{m+1} \\ &+ \frac{|\gamma|}{m+1}|x|^{m+1} + |\gamma\alpha||x(t-h)|^{m+1} + |\lambda\beta|\,\|p\|\,|x|^{2n+1} \\ &+ |\lambda\gamma|\,\|p\|\,|x(t-h)|^m]. \end{aligned}$$

(To shorten computation in the proof of the next lemma, if $m+1$ is even, the third to the last term is

$$(*) \qquad -\gamma\alpha|x(t-h)|^{m+1} \text{ .)}$$

Next, define the function

$$W(t) = 2\lambda\Big[\big(\frac{|\gamma|m}{m+1} + |\gamma\alpha|\big)\int_{t-h}^{t}|x(s)|^{m+1}ds + |\gamma\lambda|\,\|p\|\int_{t-h}^{t}|x(s)|^{m}ds + \frac{|\alpha\beta|}{2n+2}\int_{t-h}^{t}x^{2n+2}(s)ds\Big].$$

with derivative

$$W'(t) = 2\lambda\Big[\big(\frac{|\gamma|m}{m+1} + |\gamma\alpha|\big)(|x|^{m+1} - |x(t-h)|^{m+1} + |\gamma\lambda|\,\|p\|\,(|x|^{m} - |x(t-h)|^{m}) + \frac{|\alpha\beta|}{2n+2}(x^{2n+2} - x^{2n+2}(t-h))\Big]. \tag{29}$$

If we now form $V+W$, then the derivative has the first term in (26), while (29) added to Q yields a quantity

$$\overline{Q} \le 2\lambda\Big[\Big\{\beta + |\alpha\beta|\big(\frac{2n+1}{2n+2}\big) + \frac{|\alpha\beta|}{2n+2}\Big\}x^{2n+2} + \left(\frac{|\gamma|}{m+1} + \frac{m|\gamma|}{m+1} + |\gamma\alpha|\right)|x|^{m+1} + |\lambda\beta|\,\|p\|\,|x|^{2n+1} + |\gamma\lambda|\,\|p\|\,|x|^{m}\Big]$$

or

$$\overline{Q} \le 2\lambda[\{\beta + |\alpha\beta|\}x^{2n+2} + (|\gamma| + |\gamma\alpha|)|x|^{m+1} + |\lambda\beta|\,\|p\|\,|x^{2n+1}| + |\gamma\lambda|\,\|p\|\,|x|^{m}]. \tag{30}$$

Since $m < 2n$, and since $\beta < 0$ and $|\alpha| < 1$, there is an $M > 0$ with

$$(V+W)' \le 2\lambda\left[-\frac{|\beta|(1-|\alpha|)}{2}x^{2n+2} + M\right]. \tag{31}$$

But $x \in \mathcal{P}_T$ yields $V+W \in \mathcal{P}_T$ so

$$0 = V(T) + W(T) - V(0) - W(0) \le 2\lambda\Big[-\frac{|\beta|(1-|\alpha|)}{2}\int_0^T x^{2n+2}(s)ds + MT\Big].$$

As $\lambda > 0$ we get (28). This proves Lemma 2.

LEMMA 3. If $C_{st} \le 0$, if $\beta < 0$, if $\gamma\alpha > 0$, if $2n+2 = m+1$, and if $|\beta| > |\gamma|$, then (28) holds.

PROOF. Proceeding with V and Q, as before in the proof of Lemma 2 and using (*) in that proof, we get

$$\begin{aligned} Q \le 2\lambda\Big[&\beta x^{2n+2} + |\alpha\beta|\big(\frac{2n+1}{2n+2}\big)x^{2n+2} \\ &+ \frac{|\alpha\beta|}{2n+2}x^{2n+2}(t-h) + |\gamma|\big(\frac{2n+1}{2n+2}\big)x^{2n+2}(t-h) \\ &+ \frac{|\gamma|}{2n+2}x^{2n+2} - \gamma\alpha x^{2n+2}(t-h) + |\lambda\beta|\,\|p\|\,|x|^{2n+1} \\ &+ |\gamma\lambda|\,\|p\|\,|x(t-h)|^{2n+1}\Big]. \end{aligned}$$

Define a function

$$\begin{aligned} Z(t) = 2\lambda\Big[&\Big\{\frac{|\alpha\beta|}{2n+2} - \gamma\alpha + |\gamma|\big(\frac{2n+1}{2n+2}\big)\Big\}\int_{t-h}^{t} x^{2n+2}(s)ds \\ &+|\gamma\lambda|\,\|p\|\int_{t-h}^{t}|x(s)|^{2n+1}ds\Big] \end{aligned}$$

with derivative

$$\begin{aligned} Z'(t) = 2\lambda\Big[&\Big\{\frac{|\alpha\beta|}{2n+2} - \gamma\alpha + |\gamma|\big(\frac{2n+1}{2n+2}\big)\Big\}\{x^{2n+2}(t) - x^{2n+2}(t-h)\} \\ &+ |\gamma\lambda|\,\|p\|\,\big(|x|^{2n+1} - |x(t-h)|^{2n+1}\big)\Big]. \end{aligned}$$

Thus,

$$\begin{aligned} Q + Z' \le 2\lambda\big[&(\beta + |\alpha\beta| - \gamma\alpha + |\gamma|)x^{2n+2} \\ &+(|\lambda\beta|\,\|p\| + |\gamma\lambda|\,\|p\|)|x|^{2n+1}\big]. \end{aligned} \tag{32}$$

Now,

$$\begin{aligned} \beta + |\alpha\beta| + |\gamma| - \gamma\alpha &= -|\beta|(1-|\alpha|) + |\gamma|(1-|\alpha|) \\ &= (-|\beta| + |\gamma|)(1-|\alpha|) < 0. \end{aligned}$$

This means that there are positive constants U and R with

$$(V+Z)' \leq 2\lambda[-Ux^{2n+2}+R]$$

and (28) will now follow, proving Lemma 3.

The following theorem is a combination of theorems of Schaefer and Krasnoselskii (cf. [12; pp. 29–31]). It is in the unpublished manuscript [1] and may be stated as follows.

THEOREM 4. Let $(\mathcal{B}, \|\cdot\|)$ be a Banach space, $A, B : \mathcal{B} \to \mathcal{B}$, B a contraction with contraction constant $\alpha < 1$, and A continuous with A mapping bounded sets into compact sets. Either

(i) $x = \lambda B(x/\lambda) + \lambda Ax$ has a solution for $\lambda = 1$, or

(ii) the set of all such solutions, $0 < \lambda < 1$, is unbounded.

THEOREM 5. Let (20), (21) hold, $C_s \leq 0$, $C_{st} \leq 0$, $|\alpha| < 1$, and suppose that (28) holds for any T-periodic solution x of (25). Then (25) has a T-periodic solution for $\lambda = 1$.

PROOF. In the notation of Theorem 4,

$$(Bx)(t) = \alpha x(t-h)$$

and

$$(Ax)(t) = \int_t^\infty (\beta x^{2n+1}(s) + \gamma x^m(s-h))C(t,s)ds + p(t),$$

while $(\mathcal{B}, \|\cdot\|)$ is $(\mathcal{P}_T, \|\cdot\|)$, the Banach space of continuous T-periodic functions with the supremum norm. By the conditions in (20)–(21) we can show that A is continuous. Computations will show that bounded sets are mapped into equicontinuous sets. We will use (28) to show that (ii) of Theorem 4 can not hold, and that will complete the proof. We give the details for $m = 2n+1$. The case is similar when $m < 2n+1$.

From (25) we get (integrating by parts)

$$\begin{aligned}(x - \alpha x(t-h) - \lambda p)^2 &\leq \left[\int_t^\infty (\beta x^{2n+1}(s) + \gamma x^{2n+1}(s-h))C(t,s)ds\right]^2 \\ &= \Big[-C(t,s)\int_s^t (\beta x^{2n+1}(u) + \gamma x^{2n+1}(u-h))du\Big|_t^\infty \\ &\quad - \int_t^\infty -C_s(t,s)\int_s^t (\beta x^{2n+1}(u) + \gamma x^{2n+1}(u-h)du\, ds\Big]^2\end{aligned}$$

(The first term on the right is zero by (21) if $x \in \mathcal{P}_T$.)

$$
\begin{aligned}
&\leq \left[\int_t^\infty \sqrt{-C_s(t,s)}\sqrt{-C_s(t,s)}\int_s^t (\beta x^{2n+1}(u) + \gamma x^{2n+1}(u-h))du\, ds\right]^2 \\
&\leq \int_t^\infty -C_s(t,s)ds \int_t^\infty -C_s(t,s)\left(\int_s^t (\beta x^{2n+1}(u) + \gamma x^{2n+1}(u-h))du\right)^2 ds \\
&\leq C(t,t)\int_t^\infty -C_s(t,s)(s-t)\int_t^s K(x^{2n+2}(u) + x^{2n+2}(u-h) + M)du\, ds
\end{aligned}
$$

(for some K and $M > 0$, using the Schwarz inequality). Since $x \in \mathcal{P}_T$ and (28) holds, we can find constants μ, η, and Y with

$$C(t,t)\int_t^\infty -C_s(t,s)\mu(s-t+\eta)^2 ds \leq Y^2$$

by (21). This means that

$$|x - \alpha x(t-h) - \lambda p| \leq Y$$

and

$$
\begin{aligned}
|x(t)| &\leq Y + \|p\| + |\alpha||x(t-h)| \\
&=: \delta + |\alpha||x(t-h)|.
\end{aligned}
$$

If $0 \leq t \leq T$ and n is a positive integer, then

$$
\begin{aligned}
|x(t+nh)| &\leq \delta + |\alpha||x(t+(n-1)h)| \\
&\leq \delta + |\alpha|(\delta + |\alpha||x(t+(n-2)h)|) \\
&\leq \delta(1+|\alpha|) + \alpha^2(\delta + |\alpha||x(t+(n-3)h)|) \\
&\leq \cdots \\
&\leq \delta\sum_{n=1}^{\infty}|\alpha|^{n-1} + |\alpha|^{n-1}|x(t-h)| \\
&\to \delta/(1-|\alpha|) \text{ as } n \to \infty.
\end{aligned}
$$

This gives the a priori bound and proves the theorem.

5. The equation (1). We conclude with some remarks about

$$x' = ax + \alpha x'(t-h) - q(t, x, x(t-h)). \tag{1}$$

Our results have all been of a global nature, while (1) is usually of interest for only a certain range of x values, namely $x > 0$, since (1) is a population problem. In addition, if we seek periodic solutions of (1), then under common assumptions one can be found by inspection as an equilibrium solution; this means that the struggle to prove that there is a periodic solution may simply yield an obvious one.

Equation (18) is quite general and, in that form, the infinite integral seems to be unavoidable. But (1) is special and, if one is interested only in periodic solutions, then (1) can be converted to a neutral integral equation of finite delay type as follows. We wrote (1) as

$$[(x - \alpha x(t-h))e^{-at}]' = [a\alpha x(t-h) - q(t, x, x(t-h))]e^{-at}.$$

If $x \in \mathcal{P}_T$ solves that equation, integrate from $t-T$ to t and use $x(t+T) = x(t)$ to obtain

$$\begin{aligned}(x(t) - \alpha x(t-h))e^{-at} &- (x(t) - \alpha x(t-h))e^{-a(t-T)} \\ &= \int_{t-T}^{t} [a\alpha x(s-h) - q(s, x(s), x(s-h)]e^{-as} ds\end{aligned}$$

so that

$$\begin{aligned}x(t) &= \alpha x(t-h) \\ &\quad + \frac{1}{1-e^{-aT}} \int_{t-T}^{t} [a\alpha x(s-h) - q(s, x(s), x(s-h))]e^{a(t-s)} ds.\end{aligned} \tag{33}$$

It is now possible to stipulate conditions so that we can find numbers $0 < c < d$ for which the complete metric space of continuous T-periodic functions ϕ with $c \le \phi(t) \le d$ will be mapped into itself by

$$(P\phi)(t) = \alpha\phi(t-h) + \frac{1}{1-e^{-aT}} \int_{t-T}^{t} [a\alpha\phi(s-h) - q(s, \phi(s), \phi(s-h))]e^{a(t-s)} ds$$

and to obtain a fixed point by Schauder's theorem. This periodic function will satisfy (1).

Thus, (1) is special in several ways and the theory we have constructed is not suited to it. Good discussions of periodic solutions for it are found in [5] and [8].

On the other hand, non-periodic solutions of (1I) bear no relation to (33) and the integral representation of (1) will be of advanced type. All of our theory must then be rewritten as local results if it is to apply to (1).

References

1. Burton, T.A. and Kirk, C., A fixed point theorem of Krasnoselskii-Schaefer type, submitted for publication.
2. Coddington, E.A. and Levinson, N., *Theory of Ordinary Differential Equations*, McGraw-Hill, New York, 1955.
3. Driver, R.D., Existence and continuous dependence of solutions of a neutral functional-differential equation, *Archive Rat. Mech. Anal.* **19**(1965), 149–186.
4. El'sgol'ts, L.E., *Introduction to the Theory of Differential Equations with Deviating Arguments*, Holden-Day, San Francisco, 1966.
5. Gopalsamy, K., *Stability and Oscillations in Delay Differential Equations of Population Dynamics*, Kluwer, Dordrecht, 1992.
6. Gopalsamy, K. and Zhang, B.G., On a neutral delay logistic equation, *Dynamics Stability Systems* **2**(1988), 183–195.
7. Hewitt, E. and Stromberg, K., *Real and Abstract Analysis*, Springer, Berlin, 1971.
8. Kuang, Y., *Delay Differential Equations with Applications to Population Dynamics*, Academic Press, Boston, 1993.
9. Kuang, Y., Global Stability in one or two species neutral delay population models, *Canadian Appl. Math. Quart.* **1**(1993), 23–45.
10. Kuang, Y., On neutral delay logistic Gause-type predator-prey systems, *Dynamics Stability Systems* **6**(1991), 173–189.
11. Lakshmikantham, V. and Leela, S., *Differential and Integral Inequalities*, Vol. I, Academic Press, New York, 1969.
12. Smart, D.R., *Fixed Point Theorems*, Cambridge Univ. Press, Cambridge, 1980.