UPPER AND LOWER BOUNDS FOR LIAPUNOV FUNCTIONALS

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1 Introduction — A historical Sketch

In 1892 Liapunov presented a method for establishing stability relations for a system of ordinary differential equations, denoted by

$$x' = f(t, x), \tag{1}$$

based on the form of f rather than on solutions. It involved finding a Liapunov function, $V(t, x)$, and certain positive definite functions, $W_i(|x|)$, called wedges, such that the deriva-

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tive of V along solutions of (1), computed by $dV(t,x(t))/dt = \text{grad } V \cdot f + (\partial V/\partial t)$, satisfied relations such as

$$W_1(|x|) \le V(t,x) \le W_2(|x|), \qquad V'(t,x) \le -W_3(|x|).$$

Even though construction of V is an art, the method has enjoyed great and enduring success. Excellent accounts of the standard theory from two different points of view are found in Lakshmikantham and Leela [16] and in Yoshizawa [19].

In 1956 Krasovskii (cf. [15; pp. 143–175]) sought to extend the method to functional differential equations of the form

$$x'(t) = F(t, x_t) \tag{2}$$

where $x_t(s) = x(t+s)$ for $-h \le s \le 0$, h a positive constant. His technique was simple in the extreme. We give a brief sketch here, with more detail in the next section.

A solution of (1) is written as $x(t, t_0, x_0)$, where (t_0, x_0) is the initial condition, while a solution of (2) is written as $x(t, t_0, \phi)$, where (t_0, ϕ) is the initial condition and ϕ is an initial function. Krasovskii mechanically replaced x by x_t, (t_0, x_0) by (t_0, ϕ), $V(t,x)$ by $V(t, x_t)$, and $|x|$ by $\|x_t\|$, which is the supremum norm. Formally, it worked like a charm; there were even converse theorems when F was smooth. Moreover, both Krasovskii and his colleagues were able to construct a substantial collection of Liapunov functionals, but the functionals seemed incapable of satisfying the fundamental requirements of Krasovskii that

$$W_1(\|x_t\|) \le V(t, x_t) \text{ and } V'(t, x_t) \le -W_3(\|x_t\|).$$

Krasovskii [15; p. 151] was the first to point out this difficulty and he then proposed to substitute theorems asking that

$$W_1(|x(t)|) \le V(t, x_t) \text{ and } V'(t, x_t) \le -W_3(|x|).$$

Such inequalities could frequently be verified in the common examples. However, there was a stiff penalty. Virtually none of the results on asymptotic stability hold unless $F(t, x_t)$ is bounded for x_t bounded; moreover, the beautiful and useful results on uniform boundedness and uniform ultimate boundedness obtained for (1) are difficult to transfer to (2) unless $W_1(\|x_t\|) \le V(t, x_t) \le W_2(\|x_t\|)$. Finally, when F is not bounded for x_t bounded, then it is frequently impossible to establish $V(t, x_t) \le W_2(\|x_t\|)$, causing great difficulties.

In this paper we note that, by careful use of Sobolev's inequality, Jensen's inequality, and convexity it is possible to adjust the common Liapunov functionals so that they satisfy Krasovskii's original requirements, thereby avoiding severe boundedness restrictions on F. We illustrate this on a wide variety of examples which have been particularly difficult to treat using the standard Liapunov theory. Moreover, we show that the troublesome problem of bounding V from above can be turned to great advantage in establishing both

$$W_1(\|x_t\|) \le V(t, x_t) \quad \text{and} \quad V'(t, x_t) \le -W_3(\|x_t\|).$$

2 Introduction — Some detail

Let D be an open set in $\mathbf{R}^n$ with $0 \in D$ and let $f : [0, \infty) \times D \to \mathbf{R}^n$ be continuous. Then for each $(t_0, x_0) \in [0, \infty) \times D$ there is at least one solution $x(t, t_0, x_0)$ of

$$x' = f(t, x) \tag{1}$$

satisfying $x(t, t_0, x_0) = x_0$ and it may be continued for all future t so long as it does not approach the boundary of D.

Given a continuous function $V : [0, \infty) \times D \to [0, \infty)$. It is possible to define its derivative along a solution of (1), say $V'_{(1)}(t, x)$. Details of the derivative definition and its consequences are found in Yoshizawa [19]. In the theory of Liapunov's direct method it is supposed that there is such a V and continuous functions $W_i : [0, \infty) \to [0, \infty)$ which are strictly increasing with $W_i(0) = 0$ and which satisfy some of the following properties:

(ia) $W_1(|x|) \le V(t, x)$, $V(t, 0) = 0$,

(iia) $V'_{(1)}(t, x) \le 0$,

(iiia) $W_1(|x|) \le V(t, x) \le W_2(|x|)$,

(iva) $V'(t, x) \le -W_3(|x|)$,

(va) $V'(t, x) \le -W_3(|x|) + M$, $M > 0$, and $W_3(U) + M < 0$ for some $U > 0$,

(via) $f(t, x)$ is bounded for $|x|$ bounded.

Definitions of stability terms used below are found in ([5], [16], or [19]), for example. It is not difficult to prove the following standard stability results.

THEOREM 1. Let $V : [0, \infty) \times D \to [0, \infty)$ be continuous.

(Ia) If (ia) and (iia) hold, then $x = 0$ is stable.

(IIa) If (iia) and (iiia) hold, then $x = 0$ is uniformly stable (U.S.).

(IIIa) If (ia), (iva), and (via) hold, then $x = 0$ is asymptotically stable (A.S.).

(IVa) If (iiia) and (iva) hold, then $x = 0$ is uniformly asymptotically stable (U.A.S.).

(Va) If (iiia) and (va) hold, if $D = \mathbf{R}^n$, and if $W_1(r) \to \infty$ as $r \to \infty$, then solutions of (1) are uniform bounded (U.B.) and uniform ultimate bounded for bound B (U.U.B.).

Equation (2) is more complicated. Let $h > 0$ and let C be the Banach space of continuous functions $\phi : [-h, 0] \to \mathbf{R}^n$ with the supremum norm: $\|\phi\| \sup_{-h \le s \le 0} |\phi(s)|$, where $|\cdot|$ is any norm on $\mathbf{R}^n$. For $H > 0$, the set C_H is contained in C and is defined by $\phi \in C_H$ if $\|\phi\| < H$.

If $x : [-h, A) \to \mathbf{R}^n$ is continuous and $A > 0$, then for $t \geq 0$ the function $x_t \in C$ is defined by $x_t(s) = x(t+s)$ for $-h \leq s \leq 0$. If $F : [0, \infty) \times C_H \to \mathbf{R}^n$ is continuous then

$$x' = F(t, x_t) \tag{2}$$

is a system of functional differential equations with finite delay. It is assumed that F takes bounded sets of $[0, \infty) \times C_H$ into bounded sets in $\mathbf{R}^n$; in no sense does this mean that F must be bounded for x_t bounded.

Under these conditions, if $t_0 \geq 0$ and $\phi \in C_H$, then there is a solution $x(t_0, \phi)$ of (2) on an interval $[t_0, t_0 + \gamma)$, having value $x(t, t_0, \phi)$, with $x_{t_0}(t_0, \phi) = \phi$, and if there is an $H_1 < H$ with $|x(t, t_0, \phi)| \leq H_1$, then $\gamma = \infty$. If there is a continuous functional $V : [0, \infty) \times C_H \to [0, \infty)$, then one may define the derivative of V along solutions of (2) by

$$V'_{(2)}(t, \phi) = \limsup_{\delta \to 0^+} \big\{ V(t + \delta, x_{t+\delta}(t, \phi)) - V(t, \phi) \big\} / \delta.$$

Krasovskii's first formulation of Liapunov's direct method for (2) asked that there exists such a V satisfying some of the following properties:

(ib) $W_1(\|\phi\|) \leq V(t, \phi)$, $V(t, 0) = 0$,

(iib) $V'_{(2)}(t, \phi) \leq 0$,

(iiib) $W_1(\|\phi\|) \leq V(t, \phi) \leq W_2(\|\phi\|)$,

(ivb) $V'_{(2)}(t, \phi) \leq -W_3(\|\phi\|)$.

The integral form of (ivb) is worth noticing. To present it in its best light and to lay some ground work for subsequent results we will make use of convexity and Jensen's inequality (see Natanson [18; pp. 36–46]) and Sobolev's inequality (see Brézis [2] and our Theorem 5).

For reference we note that if $W : [a, b] \to (-\infty, \infty)$ with

$$W([t_1 + t_2]/2) \leq [W(t_1) + W(t_2)]/2$$

for any $t_1, t_2 \in [a, b]$, then W is convex downward. Moreover, if $f : [a, b] \to (-\infty, \infty)$ is increasing, then

$$F(t) = \int_a^t f(u)du$$

is convex downward. For our purposes here, Jensen's inequality may be stated as follows.

THEOREM (Jensen). Let $W : [0, \infty) \to (-\infty, \infty)$ be convex downward and let $f, p : [a, b] \to [0, \infty)$ be continuous with $\int\limits_a^b p(t)dt > 0$. Then

$$\int_a^b p(t)W(f(t))dt \geq \int_a^b p(t)dt\, W\left[\int_a^b f(t)p(t)dt \Big/ \int_a^b p(t)dt\right].$$

For $h > 0$ and a given t_0 we will frequently make use of the sequence of intervals $I_j = I_j(t_0, h)$ defined by

$$I_j(t_0, h) = [t_0 + (i-1)h, t_0 + ih]. \tag{3}$$

Since it is always possible to find a convex $W(r) \leq W_3(r)$ when r is restricted to a compact set (see Lemma 2), we may always suppose W_3 to be convex when solutions are bounded. If we integrate (ivb) along a solution $x(t) = x(t, t_0, \phi)$ from t_0 to $t > t_0 + nh$ we obtain

$$\begin{aligned} V(t, x_t) - V(t_0, \phi) &\leq -\sum_{j=1}^{n} \int_{I_j} W_3(\|x_s\|)ds \\ &\leq -\sum_{j=1}^{n} hW_3\left[\int_{I_j} \|x_s\|ds/h\right] \\ &\qquad\qquad \text{(by Jensen's inequality)} \\ &\leq -\sum_{j=1}^{n} hW_3(h\|x_{t_j}\|/h), \end{aligned}$$

or

$$V(t, x_t) \leq V(t_0, \phi) - \sum_{j=1}^{n} hW_3(\|x_{t_j}\|) \tag{ivb$'$}$$

for some $t_j \in I_j$.

In particular, then, a much more useable form of (ivb) would be

$$V(t,x_t) \le V(t_0,\phi) - \sum_{j=1}^{n} W_4(\|x_{t_0+jh}\|) \qquad (iv\mathrm{b})''$$

for $t > t_0 + nh$. And that is the form that we frequently obtain in applications.

THEOREM 2. Suppose there is a continuous $V : [0,\infty) \times C_H \to [0,\infty)$.

(Ib) If (ib) and (iib) hold, then $x = 0$ is stable.

(IIb) If (iib) and (iiib) hold, then $x = 0$ is U.S.

(IIIb) If (ib) and (ivb) hold, then $x = 0$ is A.S.

(IVb) If (iiib) and (ivb) hold, then $x = 0$ is U.A.S.

We remark that (IIIb) was neither stated nor proved by Krasovskii, but it is a corollary to a result of Burton-Hatvani [6].

Krasovskii himself was one of the first to remark that (ib) and (ivb) might seldom be satisfied. We now briefly discuss the classical prototype showing how the problem appeared to early investigators and why (ib) and (ivb) seemed unrealistic.

Let A and B be $n \times n$ matrices of constants, let $h > 0$, and consider the system

$$x'(t) = Ax(t) + Bx(t-h). \qquad (4)$$

If the characteristic roots of A all have negative real parts, then there is a unique positive definite and symmetric matrix C with $A^T C + CA = -I$. One then constructs a functional

$$V(t,x_t) = x^T(t)Cx(t) + \int_{t-h}^{t} x^T(s)Lx(s)ds \qquad (5)$$

where L is positive definite, symmetric, and to be chosen later. If L is well chosen, then for "small enough" B we have

$$V'_{(4)}(t,x_t) \le -\alpha |x(t)|^2, \quad \alpha > 0. \qquad (6)$$

We also have

$$a|x(t)|^2 \leq V(t, x_t) \leq b\|x_t\|^2 \tag{7}$$

for certain positive constants a and b. However, it would seem by inspection that

$$W_1(\|x_t\|) \leq V(t, x_t)$$

and

$$V_4'(t, x_t) \leq -W_3(\|x_t\|)$$

could not be satisfied. Thus, Krasovskii immediately introduced the following properties:

(ic) $W_1(|\phi(0)|) \leq V(t, \phi)$, $V(t, 0) = 0$,

(iic) $V_{(2)}'(t, \phi) \leq 0$,

(iiic) $W_1(|\phi(0)|) \leq V(t, \phi) \leq W_2(\|\phi\|)$,

(ivc) $V'(t, \phi) \leq -W_3(|\phi(0)|)$,

(vc) $F(t, \phi)$ is bounded for ϕ bounded,

(vic) $W_1(|\phi(0)|) \leq V(t, \phi) \leq W_2(|\phi(0)|) + W_4(|||\phi|||)$ where $|||\cdot|||$ is the L^2-norm.

Krasovskii then obtained the following result.

THEOREM 3. Suppose there is a continuous $V : [0, \infty) \times C_H \to [0, \infty)$.

(Ic) If (ic) and (iic) hold, then $x = 0$ is stable.

(IIc) If (iic) and (iiic) hold, then $x = 0$ is U.S.

(IIIc) If (iiic), (ivc), and (vc) hold then $x = 0$ is U.A.S.

(IVc) If (ivc), (vc), and (vic) hold then $x = 0$ is A.S.

REMARK. In Section 4 we show that the apparently more generous formulation in Theorem 3 can, essentially, be put in the form of Theorem 2.

In 1978 Burton [3] improved (IVc) as follows.

THEOREM 4. Suppose there is a continuous $V : [0, \infty) \times C_H \to [0, \infty)$.

(IVc)$'$ If (ivc) and (vic) hold, then $x = 0$ is U.A.S.

We remark that investigators generally did not notice that Theorem 3(IVc) required (vc) and this has produced a fair amount of confusion. See El'sgol'ts [11; p. 72], for example. It remains an important open problem to eliminate (vc) from (IIIc). This condition has blocked the study of many interesting problems for which we have beautiful Liapunov functionals.

While the reformulation (ic)–(vic) seemed to be dictated by examples, it has proved to be unfruitful in many problems of A.S., U.A.S., U.B., and U.U.B. But perhaps the most interesting difficulty is that in problems in which $F(t, \phi)$ is not bounded for ϕ bounded, then the condition $V(t, \phi) \le W(\|\phi\|)$ is almost never satisfied; and we show in this paper that this is a blessing in disguise.

We now motivate our work by returning to (4) and showing that, with a little effort, the original Krasovskii formulation in Theorem 2 can be satisfied. Using (5) we define a new functional

$$\overline{V}(t, x(\cdot)) = V(t, x_t) + V(t-h, x_{t-h}) \tag{8}$$

for $t \ge t_0 + h$, a technique introduced in Becker-Burton-Zhang [1]. It is possible to find positive constants k_i for which the following relations hold:

$$\begin{aligned}\overline{V}(t, x(\cdot)) &\ge x^T(t)Cx(t) + k_1 \int_{t-h}^{t} \left[|x(s)|^2 + |x(s-h)|^2\right] ds \\ &\ge k_2|x(t)|^2 + k_3\left[\int_{t-h}^{t} (|x(s)| + |x(s-h)|) ds\right]^2\end{aligned}$$

(by Jensen's inequality and convexity of u^2)

$$\ge k_4\left[|x(t)| + \int_{t-h}^{t} |x'(s)| ds\right]^2 \ge k_5\|x_t\|^2$$

(by Sobolev's inequality; see Theorem 5).

Next, from (6) we have $V'(t,x_t) \leq -\alpha|x(t)|^2$ and so we readily obtain $\overline{V}'(t,x(\cdot)) \leq -k_6(|x(t)|+|x'(t)|)^2$ so that for $t > t_0 + nh$ and for I_j defined by (3) we have

$$\begin{aligned}\overline{V}(t,x(\cdot)) - \overline{V}(t_0+h,x(\cdot)) &\leq -k_6 \sum_{j=2}^{n} \int_{I_j} (|x(s)|+|x'(s)|)^2 ds \\ &\leq -k_7 \sum_{j=2}^{n} \left[\int_{I_j} (|x(s)|+|x'(s)|) ds \right]^2 \\ &\leq -k_8 \sum_{j=2}^{n} \|x_{t_0+jh}\|^2\end{aligned}$$

(see Theorem 5 for details) so that

$$\overline{V}(t,x(\cdot)) \leq \overline{V}(t_0+h,x(\cdot)) - k_8 \sum_{j=2}^{n} \|x_{t_0+jh}\|^2. \tag{9}$$

Our work here is not restricted to finite delay equations but also includes problems of the form

$$x'(t) = G(t,x(s);\ \alpha \leq s \leq t) \stackrel{\text{def}}{=} G(t,x(\cdot)) \tag{10}$$

in which $\alpha \geq -\infty$ and G is defined and takes values in $\mathbf{R}^n$ for $t \geq 0$ whenever $x : [\alpha,\infty) \to \mathbf{R}^n$ is continuous and bounded for $t \leq 0$. It is supposed that for each $t_0 \geq 0$ and each bounded and continuous $\phi : [\alpha,t_0] \to \mathbf{R}^n$ there is a solution $x(t,t_0,\phi)$ of (10) on $[t_0,\beta)$ satisfying $x(t,t_0,\phi) = \phi(t)$ on $[\alpha,t_0]$ and that if the solution remains bounded then $\beta = \infty$. Conditions ensuring such behavior are found in Driver [10] and in Burton [4]. As this material is fairly complicated in the general case, we do not repeat it here. However, in the examples we use, the derivative is simply a chain rule and a Lipschitz argument.

3 Applications of Sobolev's inequality

In this section we present a form of Sobolev's inequality that will allow us to handle some Liapunov functionals for equations with $F(t, x_t)$ unbounded for x_t bounded. We will also show that Krasovskii's Theorem 3 is just a particular case of his original formulation, Theorem 2. The first lemma is essentially a form of Sobolev's inequality, tailored to our purposes. A standard proof is found in Brézis [2], but our forms are so simple that easy proofs are supplied here.

LEMMA 1. Let $\phi : [-h, 0] \to \mathbf{R}^n$ have a continuous derivative. Then

$$\min_{-h\le t\le 0} |\phi(t)| + \int_{-h}^{0} |\phi'(s)|ds \ge \|\phi\|, \tag{i}$$

$$|\phi(t)| + \int_{-h}^{0} |\phi'(s)|ds \ge \|\phi\|, \tag{ii}$$

and

$$\int_{-h}^{0} \big[|\phi(s)| + |\phi'(s)|\big] ds \ge k\|\phi\| \tag{iii}$$

where $k = \min[1, h]$.

PROOF. Choose t_1 and t_2 with

$$|\phi(t_1)| = \min_{-h\le s\le 0} |x(s)| \ \text{ and } \ |\phi(t_2)| = \|\phi\|.$$

Then

$$\phi(t_2) = \phi(t_1) + \int_{t_1}^{t_2} \phi'(s)ds$$

and so

$$\begin{aligned}|\phi(t_2)| &\le |\phi(t_1)| + \left|\int_{t_1}^{t_2} \phi'(s)ds\right| \\ &\le |\phi(t_1)| + \left|\int_{t_1}^{t_2} |\phi'(s)|ds\right| \\ &\le |\phi(t_1)| + \int_{-h}^{0} |\phi'(s)|ds\end{aligned}$$

which shows that

$$\|\phi\| = |\phi(t_2)| \le \min_{-h\le t\le 0} |\phi(t)| + \int_{-h}^{0} |\phi'(s)|ds$$

yielding (i) and, certainly (ii). To prove (iii) we have

$$\begin{aligned}&\int_{-h}^{0} \big[|\phi(s)| + |\phi'(s)|\big]ds = \int_{-h}^{0} |\phi(s)|ds + \int_{-h}^{0} |\phi'(s)|ds \\ &\ge \int_{-h}^{0} \min_{-h\le s\le 0} |\phi(s)|ds + \int_{-h}^{0} |\phi'(s)|ds \\ &= h|\phi(t_1)| + \int_{-h}^{0} |\phi'(s)|ds \\ &\ge \{\min[h,1]\}\left[|\phi(t_1)| + \int_{-h}^{0} |\phi'(s)|ds\right] \ge k\|\phi\|\end{aligned}$$

as required.

LEMMA 2. Let $W_1(r)$ and $W_2(r)$ be wedges defined for $0 \le r \le M$. Then there exists a convex wedge $W(r)$ with $W(r) \le W_1(r)$ and $W(r) \le W_2(r)$ on $[0, M]$.

PROOF. Note that

$$\begin{aligned}\overline{W}(r) &= \min\big[W_1(r), W_2(r)\big] \text{ is a wedge;} \\ \tilde{W}(r) &= \int_0^r \overline{W}(s)ds \text{ is a convex wedge;}\end{aligned}$$

and on $[0, 1]$ we have $\tilde{W}(r) \le \overline{W}(r)$. If $M > 1$, define $W(r) = \tilde{W}(r/M)$. Since $r/M \le r$ we have $\tilde{W}(r/M) < \tilde{W}(r)$ on [0,1], while on $[1, M]$ we have $W(M) = \tilde{W}(1) \le \overline{W}(1) \le \overline{W}(r)$

and since $\overline{W}$ is increasing it follows that $W(r) \leq W(M) \leq \overline{W}(r)$. Thus, $W(r) \leq \overline{W}(r) \leq \min[W_1(r), W_2(r)]$. This completes the proof.

LEMMA 3. Let $f : [0,\infty) \to [0,\infty)$ be continuous and suppose there is a sequence $\{t_n\} \uparrow \infty$ such that $\int_{t_n-h}^{t_n} f(s)ds \to \infty$. Then $\int_{t-h}^{t} f(s)ds \to \infty$ uniformly for $\tilde{t}_n \leq t \leq \tilde{t}_n + (h/2)$ where $\tilde{t}_n$ is either t_n or $[t_n + (h/2)]$.

PROOF. For any $L > 0$ there exists $N = N(L)$ such that $n \geq N$ implies that $\int_{t_n-h}^{t_n} f(s)ds > L$. Write the integral as $A_{1_n} + A_{2_n}$ where $A_{1_n} = \int_{t_n-h}^{t_n-(h/2)} f(s)ds$ and $A_{2_n} = \int_{t_n-(h/2)}^{t_n} f(s)ds$. Thus, for $n \geq N$ either $A_{1_n} > L/2$ or $A_{2_n} > L/2$. If $A_{1_n} > L/2$ then $\int_{t-h}^{t} f(s)ds > L/2$ for $t \in [t_n - h/2, t_n]$. If $A_{2_n} > L/2$, then $\int_{t-h}^{t} f(s)ds > L/2$ for $t \in [t_n, t_n + (h/2)]$. This completes the proof.

The next theorem shows how it is possible to get the inequalities involving wedges that Krasovskii used in his first formulation, when one starts from inequalities involving the derivative.

THEOREM 5. Let $V : [0,\infty) \times C_H \to [0,\infty)$ be continuous.

(I) If $x : [t_0,\infty) \to \mathbf{R}^n$ is a solution of (2) with $|x(t)| < H$ and

$$V(t,x_t) \geq W_1(|x(t)|) + W_2\left(\int_{t-h}^{t} W_3(|x(s)|)W_4(|x'(s)|)ds\right) \tag{i}$$

where W_4 is convex, then there is a convex W_5 with

$$W_5(\|x_t\|) \leq V(t,x_t). \tag{ii}$$

(II) If $x(t)$ is a solution of (2) on $[0,\infty)$ with $|x(t)| < H$ and

$$V'_{(2)}(t,x_t) \leq -W_1(|x(t)|) - W_2\left(\int_{t-h}^{t} |x'(s)|ds\right) \tag{iii}$$

then there is a convex W_3 with

$$V'_{(2)}(t, x_t) \leq -W_3(\|x_t\|). \tag{iv}$$

(III) If $x(t)$ is a solution of (2) on $[0, \infty)$ with

$$V'_{(2)}(t, x_t) \leq -W_1(|x(t)|) - W_2(|x'(t)|) \tag{v}$$

then

$$V(t, x_t) \leq V(0, x_0) - \sum_{i=1}^{k} W_3(\|x_{ih}\|) \tag{vi}$$

for $t > kh$ and some convex W_3.

PROOF. To prove (I) we let $x(t) = x(t, t_0, \phi)$ and consider two cases. If $t \geq t_0$ and $|x(t)| \geq (1/2)\|x_t\|$, then

$$V(t, x_t) \geq W_1((1/2)\|x_t\|) \stackrel{\text{def}}{=} \tilde{W}_5(\|x_t\|).$$

If $|x(t)| < (1/2)\|x_t\|$ then there are points $t_1, t_2 \in [t-h, t]$ with $|x(t_1)| = \|x_t\|$, $|x(t_2)| = (1/2)\|x_t\|$, and $|x(s)| \geq (1/2)\|x_t\|$ for all s between t_1 and t_2. Thus,

$$\begin{aligned} W_2\left(\int_{t-h}^{t} W_3(|x(s)|)W_4(|x'(s)|)ds\right) &\geq W_2\left(\left|\int_{t_1}^{t_2} W_3(|x(s)|)W_4(|x'(s)|)ds\right|\right) \\ &\geq W_2\left(W_3((1/2)\|x_t\|)\left|\int_{t_1}^{t_2} W_4(|x'(s)|)ds\right|\right) \\ &\geq W_2\left(W_3((1/2)\|x_t\|)hW_4\left(\left|\int_{t_1}^{t_2} |x'(s)|ds\right|/h\right)\right) \\ &\geq W_2(W_3((1/2)\|x_t\|)hW_4((1/2)\|x_t\|)) \stackrel{\text{def}}{=} W_5(\|x_t\|). \end{aligned}$$

We take $\overline{\overline{W}}(r) = \min[\tilde{W}_5(r), \overline{W}_5(r)]$. Then find $W_5(r) \leq \overline{\overline{W}}_5(r)$ with W_5 convex to complete the proof of (I).

To prove (II) we first note that $\int_{t-h}^{t} |x'(s)|ds$ is bounded; otherwise, apply Lemma 3, integrate (iii) from 0 to t, and conclude that $V(t,x_t) \to -\infty$. Next, apply Lemma 2 and conclude that there is a convex W_3 with

$$\begin{aligned}
W_1(|x(t)|) + W_2\left(\int_{t-h}^{t} |x'(s)|ds\right) &\geq W_3(|x(t)|) + W_3\left(\int_{t-h}^{t} |x'(s)|ds\right) \\
&\geq 2W_3\left((1/2)\left(|x(t)| + \int_{t-h}^{t} |x'(s)|ds\right)\right) \\
&\geq 2W_3((1/2)\|x_t\|)
\end{aligned}$$

by Lemma 1.

To prove (III) we integrate (v) so that for $t > nh$ we have

$$\begin{aligned}
V(t,x_t) &\leq V(0,x_0) - \int_0^t \big[W_1(|x(s)|) + W_2(|x'(s)|)\big]ds \\
&\leq V(0,x_0) - \sum_{j=1}^{n} \int_{I_j} \big[W_1(|x(s)|) + W_2(|x'(s)|)\big]ds \\
&\leq V(0,x_0) - \sum_{j=1}^{n} h\left[W_1\left(\int_{I_j} (|x(s)|)/h\right) + W_2\left(\int_{I_j} (|x'(s)|)/h\right)\right]
\end{aligned}$$

by Jensen's inequality. Clearly, these integrals are bounded since $V \geq 0$. Hence, by Lemma 2 there is a convex W_3 for which the result holds. This completes the proof.

We now give two propositions linking Krasovskii's second formulation to his first one. The work in Section 2 of showing how to transform a Liapunov functional to a better form depended strongly on the form of the functional. But if $F(t,\phi)$ is bounded for $\phi \in C_H$, then the form takes on less importance. A weaker form of this result is found in [4; p. 258].

PROPOSITION 1. If there is a continuous $V : [0,\infty) \times C_H \to [0,\infty)$ with

(i) $W_1(|x(t)|) \leq V(t,x_t)$, $V(t,0) = 0$,

(ii) $V'_{(2)}(t,x_t) \leq -W_2(|x(t)|)$,

(iii) $|F(t,x_t)| \le M$ for $x_t \in C_H$, $M \ge 1$,

then there is a W_3 with

(iv) $V'_{(2)}(t,x_t) \le -(1/2)\big[W_2(|x(t)|) + W_2(|x(t)|)|x'(t)|/M\big]$ and for $x(t) = x(t,t_0,\phi)$ with $t > t_0 + kh$ we have

(v) $V(t,x_t) - V(t_0,\phi) \le -(1/2M)\sum_{j=1}^{k} W_3(\|x_{t_0+jh}\|)$.

PROOF. Since $|F(t,x_t)| \le M$, (iv) is clear. We let $|x(t)| < H$, take $I_j = [t_0 + (j-1)h, t_0 + jh]$, and $\|x\|_j = \|x_{t_0+jh}\|$. Then

$$\begin{aligned} V(t,x_t) - V(t_0,\phi) &\le -(1/2M)\sum_{j=1}^{k}\int_{I_j}\big[W_2(|x(s)|) + W_2(|x(s)|)|x'(s)|\big]ds \\ &\le -(1/2M)\sum_{j=1}^{k}\left[h\min_{s\in I_j} W_2(|x(s)|) + \int_{I_j} W_2(|x(s)|)|x'(s)|ds\right]. \end{aligned}$$

If $\min_{s\in I_j}|x(s)| \ge (1/2)\|x\|_j$, then the j-th term in the series is bounded below by

$$hW_2((1/2)\|x\|_j).$$

If $\min_{s\in I_j}|x(s)| < (1/2)\|x\|_j$, then there are points $t_1, t_2 \in I_j$ with

$$|x(t_1)| = \|x\|_j, \quad |x(t_2)| = (1/2)\|x\|_j, \quad |x(s)| \ge (1/2)\|x\|_j$$

for s between t_1 and t_2. Thus,

$$\begin{aligned} \int_{I_j} W_2(|x(s)|)|x'(s)|ds &> W_2((1/2)\|x\|_j)\left|\int_{t_1}^{t_2}|x'(s)|ds\right| \\ &\ge W_2((1/2)\|x\|_j)(1/2)\|x\|_j. \end{aligned}$$

If

$$W_3(r) = \min[hW_2(r/2), W_2(r/2)r/2],$$

then

$$V(t,x_t) - V(t_0,\phi) \le -(1/2M)\sum_{j=1}^{k} W_3(\|x\|_j).$$

This completes the proof.

PROPOSITION 2. If $V : [0,\infty) \times C_H \to [0,\infty)$ is continuous, if $x(t) = x(t,t_0,\phi)$ is a solution of (2) satisfying $|x(t)| < H$, if

$$V(t,x_t) \ge W_1(|x(t)|) + W_2\left(\int_{t-h}^{t} W_3(|x(s)|)ds\right) \tag{i}$$

and if

$$|F(t,x_t)| \le M \quad \text{for} \quad x_t \in C_H, \tag{ii}$$

then there is a W_4 with

$$V(t,x_t) \ge W_4(\|x_t\|). \tag{iii}$$

To prove the result we write

$$V(t,x_t) \ge W_1(|x(t)|) + W_2\left(\int_{t-h}^{t} W_3(|x(s)|)(|x'(s)|/M)ds\right)$$

and apply Theorem 5(I).

4 Some motivating examples

In this section we present examples which have proved to be particularly troublesome when approached with the standard theory. We make extensive use of Theorem 5 in Section 3 to show how the Liapunov functionals used in the examples satisfy Krasovskii's first formulation.

EXAMPLE A. Consider the scalar equation

$$x'(t) = b(t)x(t/2) - a(t)x(t) \tag{A1}$$

in which $a, b : [0, \infty) \to (-\infty, \infty)$ are continuous. If there are constants θ_1, θ_2 with

$$0 < \theta_1 < 1, \quad \theta_2 > 0, \tag{A2}$$

$$|b(t)| - (\theta_1/2)a(t/2) \leq -\theta_2|b(t)|, \tag{A3}$$

and if

$$\int_0^\infty a(t)dt = \infty \tag{A4}$$

then for

$$V(t, x(\cdot)) = |x(t)| + \theta_1 \int_{t/2}^t a(s)|x(s)|ds$$

we have

$$V(t, x(\cdot)) \leq -\alpha\big[a(t)|x(t)| + |x'(t)|\big], \quad \alpha > 0 \tag{A5}$$

and $x = 0$ is A.S.

Moreover, if $a(t) > 0$ then the functional

$$\overline{V}(t, x(\cdot)) = V(t, x(\cdot)) + V(t/2, x(\cdot))$$

satisfies

$$\overline{V}'(t, x(\cdot)) \leq -\alpha a(t)|x(t)| \tag{A6}$$

and

$$\overline{V}(t, x(\cdot)) \geq \beta\big[\{|x'(t)|/a(t)\} + |x(t)|\big], \quad \beta > 0, \quad a(t) > b(t), \tag{A7}$$

so that for

$$\tilde{V}(t, x(\cdot)) = \overline{V}(t, x(\cdot)) + (1/2)V^2(t, x(\cdot))$$

we have

$$\tilde{V}(t, x(\cdot)) \leq -\alpha a(t)|x(t)| - \alpha\beta|x(t)|\,|x'(t)|. \tag{A8}$$

Moreover, if $\|x\|^J$ denotes the supremum of $|x|$ on an interval J, then

$$\overline{V}(t, x(\cdot)) \geq \overline{\mu}\|x\|^{[t/2,t]}, \quad \overline{\mu} > 0; \tag{A9}$$

and if $J_1 = [t_0, 2t_0],\ J_2 = [2t_0, 4t_0], \ldots$ then for $a(t) \geq a_0 > 0$ it follows that

$$\overline{V}(t, x(\cdot)) \leq V(t_0, x(\cdot)) - \overline{\alpha}\sum_{j=1}^{k}\|x\|^{J_j} \tag{A10}$$

and a similar expression is valid for $\overline{V}$.

PROOF. We have

$$\begin{aligned} V'(t, x(\cdot)) &\leq |b(t)|\,|x(t/2)| - a(t)|x(t)| + \theta_1 a(t)|x(t)| - (\theta_1/2)a(t/2)|x(t/2)| \\ &\leq (\theta_1 - 1)a(t)|x(t)| - \theta_2|b(t)|\,|x(t/2)| \end{aligned}$$

so that if $2\alpha = \min[\theta_2, 1 - \theta_1]$ then (A5) holds. A.S. now follows from Theorem 6 of Section 5.

To prove (A9) we note that

$$\begin{aligned} \overline{V}(t, x(\cdot)) &\geq |x(t)| + \theta_1\int_{t/2}^{t} a(s)|x(s)|ds + \theta_1\int_{t/4}^{t/2} a(s)|x(s)|ds \\ &\geq |x(t)| + (\theta_1/2)\int_{t/2}^{t}\big[a(s)|x(s)| + a(s/2)|x(s/2)|\big]ds \\ &\geq |x(t)| + \mu\int_{t/2}^{t}|x'(s)|ds, \quad \mu > 0. \end{aligned}$$

Then (A9) follows from Theorem 5.

When $a(t) \geq a_0$, then (A5) yields

$$V'(t, x(\cdot)) \leq -\overline{\alpha}\big[|x(t)| + |x'(t)|\big]$$

so that for t past J_k then

$$V'(t, x(\cdot)) \le V(t_0, x(\cdot)) - \overline{\alpha} \sum_{j=1}^{k} \int_{I_j} \big[|x(s)| + |x'(s)|\big] ds$$

and (A10) will follow from Lemma 1(i). This completes the proof.

REMARK. Example A is notable because:

(a) $G(t, x(\cdot))$ is not bounded for $x(\cdot)$ bounded.

(b) The delay is unbounded.

(c) The derivative of V can vanish over arbitrarily long time intervals; yet, we conclude A.S.

(d) Inequalities (A9) and (A10) are the ones which Krasovskii needed in his first formulation.

The next example concerns (A1) again and it requires much more of $a(t)$. But it more clearly illustrates how unboundedness of the Liapunov functional can be used to great advantage.

EXAMPLE B. Consider the scalar equation

$$x'(t) = b(t)x(t/2) - a(t)x(t) \tag{B1}$$

with $a, b : [0, \infty) \to (-\infty, \infty)$ being continuous. If there is a $\theta \in (0, 1)$ with

$$a'(t) + (\theta - 1)a^2(t) \le -\alpha \le 0, \tag{B2}$$

$$b^2(t) - (\theta/2)a^2(t/2) \le -\beta \le 0. \tag{B3}$$

$$\alpha + \beta > 0, \quad a(t) \ge a_1 > 0, \tag{B4}$$

then $x=0$ is A.S.

Moreover, if

$$V(t,x(\cdot)) = a(t)x^2(t) + \theta\int_{t/2}^{t} a^2(s)x^2(s)ds$$

then

$$V'(t,x(\cdot)) \le -\alpha x^2(t) - \beta x^2(t/2), \tag{B5}$$

$$V(t,x(\cdot)) \ge \gamma\int_{t}^{2t} |x'(s)|^2 ds, \quad \gamma > 0, \tag{B6}$$

and if $\{J_n\}$ is the sequence in Example A then for t past J_n we have

$$V(t,x(\cdot)) - V(t_0,x(\cdot)) \quad \le -\big[\gamma/V(t_0,x(\cdot))\big]\sum_{j=2}^{n}\left(\int_{J_j}[\alpha x^2(s)+\beta x^2(s/2)]^{1/2}|x'(s)|ds\right)^2. \tag{B7}$$

PROOF. A calculation yields (B5). To prove (B6) note that

$$|x'(t)|^2 \le 2[a^2(t)x^2(t) + b^2(t)x^2(t/2)]$$

while

$$\begin{aligned} V(t,x(\cdot)) &\ge \theta\int_{t/2}^{t} a^2(s)x^2(s)ds = \theta\int_{t}^{2t} a^2(s/2)x^2(s/2)ds \\ &\ge \int_{t}^{2t} b^2(s)x^2(s/2)ds. \end{aligned}$$

Since $V'(t,x(\cdot)) \le 0$, then

$$V(t,x(\cdot)) \ge V(2t,x(\cdot)) \ge \theta\int_{t}^{2t} a^2(s)x^2(s)ds$$

so that

$$2V(t,x(\cdot)) \geq \int_t^{2t} \left[\theta a^2(s)x^2(s) + 2b^2(s)x^2(s/2)\right] ds$$
$$\geq 2\gamma \int_t^{2t} |x'(s)|^2 ds$$

which yields (B6). Thus,

$$V(t_0,x(\cdot)) \geq V(jt_0,x(\cdot)) \geq \gamma \int_{jt_0}^{2jt_0} |x'(s)|^2 ds$$

and so

$$1 \geq \gamma \int_{J_j} |x'(s)|^2 ds / V(t_0,x(\cdot)).$$

This, together with (B5) yields

$$\begin{aligned} &V(t,x(\cdot)) - V(t_0,x(\cdot)) \\ &\quad \leq -\sum_{j=1}^{n} \int_{J_j} [\alpha x^2(s) + \beta x^2(s/2)] ds \\ &\quad \leq -\sum_{j=2}^{n} \gamma \int_{J_j} [\alpha x^2(s) + \beta x^2(s/2)] ds \int_{J_j} |x'(s)|^2 ds / V(t_0,x(\cdot)) \end{aligned}$$

from which we obtain (B7) by Schwarz' inequality.

The A.S. will follow from V, (B5), (B6) and Theorem 7 of Section 5. This completes the proof.

EXAMPLE C. Let $h > 0$ and consider the scalar equation

$$x'(t) = -(t+h+1)x(t) + tx(t-h). \tag{C1}$$

Then $x = 0$ is A.S. Also if

$$V(t,x_t) = |x(t)| + \int_{t-h}^{t} (s+h)|x(s)| ds \tag{C2}$$

then

$$V'(t,x_t) \le -|x(t)| \tag{C3}$$

and

$$V(t,x_t) \ge \gamma \|x_{t+h}\|, \quad \gamma > 0. \tag{C4}$$

If

$$\overline{V}(t,x(\cdot)) = V(t,x_t) + V(t-h,x_{t-h}),$$

then

$$\overline{V}'(t,x(\cdot)) \le -\big[|x(t)| + |x(t-h)|\big], \tag{C5}$$

$$\overline{V}(t,x(\cdot)) \ge |x'(t)|/(t+h+1), \tag{C6}$$

and for

$$\tilde{V}(t,x(\cdot)) = \overline{V}(t,x(\cdot)) + (1/2)\overline{V}^2(t,x(\cdot))$$

we have $\tilde{V}(t,x(\cdot)) \ge \gamma\|x_t\|$ and

$$\tilde{V}'(t,x(\cdot)) \le -|x(t)| - \big[|x(t)|\,|x'(t)|/(t+h+1)\big]. \tag{C7}$$

PROOF. A calculation yields

$$\begin{aligned} V'(t,x_t) &\le -(t+h+1)|x(t)| + t|x(t-h)| \\ &\quad + (t+h)|x(t)| - t|x(t-h)| = -|x(t)|. \end{aligned}$$

Also, since $V' \le 0$ we have

$$V(t+h,x_{t+h}) \ge \int_t^{t+h} (s+h)|x(s)|ds$$

and

$$V(t,x_t) \geq \int_{t-h}^{t} (s+h)|x(s)|ds = \int_{t}^{t+h} s|x(s-h)|ds$$

so that

$$(2+(1/h))V(t,x_t) \geq \int_{t}^{t+h} |x'(s)|ds + |x(t)| \geq \|x_{t+h}\|.$$

Then (C4) will follow from Lemma 1(ii). Also, (C5)–(C7) are obvious.

A simple Razumikhin argument with x^2 yields U.S. The asymptotic stability will now follow from Theorem 8.

REMARK. Example C is notable because:

(a) $F(t,x_t)$ is not bounded.

(b) The growth of V enables us to get $|x'|$ in the derivative of $\tilde{V}$.

(c) Inequalities (C4) and (C7) give properties needed in Krasovskii's first formulation.

The next example, along with Theorem 6, show that the techniques are also effective on nonlinear equations.

Example D. Consider the scalar equation

$$x'(t) = b(t)x^3(t-h) - C(t)x^3(t) \tag{D1}$$

with

$$C(t) \notin L^1[0,\infty) \tag{D2}$$

and

$$|b(t)| - \theta C(t-h) \leq 0, \quad 0 < \theta < 1. \tag{D3}$$

Then $x = 0$ is A.S. Also, for

$$V(t, x_t) = |x(t)| + \theta \int_{t-h}^{t} C(s)|x^3(s)|ds$$

and

$$\overline{V}(t, x(\cdot)) = V(t, x_t) + V(t-h, x_{t-h})$$

then

$$\overline{V}(t, x(\cdot)) \geq \alpha \|x_t\|, \quad \alpha > 0, \tag{D4}$$

and

$$\overline{V}'(t, x(\cdot)) \leq \big[\theta(\theta-1)/2\big]|x'(t)| + \big[(\theta-1)/2\big]C(t)|x^3(t)|. \tag{D5}$$

PROOF. We have

$$\begin{aligned} V'(t, x_t) &\leq |b(t)|\,|x^3(t-h)| - C(t)|x^3(t)| + \theta C(t)|x^3(t)| \\ &\quad - \theta C(t-h)|x^3(t-h)| \\ &\leq (\theta-1)C(t)|x^3(t)| + \big[|b(t)| - \theta C(t-h)\big]|x^3(t-h)| \\ &\leq (\theta-1)C(t)|x^3(t)|. \end{aligned}$$

Hence,

$$\overline{V}'(t, x(\cdot)) \leq (\theta-1)\big[C(t)|x^3(t)| + C(t-h)|x^3(t-h)|\big]$$

so that (D5) holds. Next,

$$\begin{aligned} V(t-h, x_{t-h}) &= |x(t-h)| + \theta \int_{t-2h}^{t-h} C(s)|x^3(s)|ds \\ &= |x(t-h)| + \theta \int_{t-h}^{t} C(s-h)|x^3(s-h)|ds \end{aligned}$$

so that

$$\overline{V}(t,x(\cdot)) \geq |x(t)| + \theta \int_{t-h}^{t} \big[C(s)|x^3(s)| + C(s-h)|x^3(s-h)|\big]ds$$

and (D4) will follow from Lemma 1. The proof of A.S. will then follow from Theorem 6.

EXAMPLE D (revisited). Consider Example D once more with $h = h(t) \geq 0$ and suppose there are positive constants M and θ with

$$1 - h'(t) > 0, \quad \big[1 + (1/M)\big]|b(t)| \leq (1 - h'(t))\theta C(t-h), \tag{D6}$$

and

$$M\theta < M - 1, \quad \int_0^\infty C(t)dt = \infty. \tag{D7}$$

Then $x = 0$ is A.S. Also, if

$$V(t,x_t) = |x(t)| + \int_{t-h(t)}^{t} \theta C(s)|x^3(s)|ds$$

then

$$V'(t,x_t) \leq -\eta_1(t)|x^3(t)| - \eta_2|x'(t)| \tag{D8}$$

where $\eta_1(t) = \lambda C(t)$, $\lambda > 0$, and η_2 is a positive constant. The conclusion of A.S. follows from Theorem 6.

EXAMPLE E. Consider the scalar equation

$$x'(t) = -a(t)x(t) + \int_0^t C(t-s)x(s)ds \tag{E1}$$

with a and C continuous on $[0,\infty)$,

$$-a(t) + \int_0^\infty |C(u)|du \leq -\alpha a(t) \leq -\beta < 0, \tag{E2}$$

and

$$\int_t^\infty |C(u)|du \geq K|C(t)|, \quad K > 0. \tag{E3}$$

Then $x = 0$ is A.S. Also, if

$$V(t, x(\cdot)) = |x(t)| + \int_0^t \int_{t-s}^{\infty} |C(u)|du|x(s)|ds$$

then

$$V'(t, x(\cdot)) \le -\alpha a(t)|x| \le -\beta|x| \tag{E4}$$

so that any solution is $L^2[0, \infty)$,

$$V(t_0, x(\cdot)) \ge V(t, x(\cdot)) \ge M|x'(t)|/a(t), \quad M > 0, \tag{E5}$$

and for

$$\overline{V}(t, x(\cdot)) = V(t, x_t) + \left(\frac{1}{2}\right) V^2(t, x_t)$$

it follows that

$$\overline{V}'(t, x(\cdot)) \le -\beta|x(t)| - \alpha M|x(t)|\,|x'(t)|. \tag{E6}$$

PROOF. We have

$$\begin{aligned} V'(t, x(\cdot)) \le -a(t)|x(t)| &+ \int_0^t |C(t-s)|\,|x(s)|ds \\ &- \int_0^t |C(t-s)|\,|x(s)|ds \\ &+ \int_0^{\infty} |C(u)|du|x(t)| \le -\alpha a(t)|x(t)| \end{aligned}$$

so that (E4) holds. Next, note that

$$\begin{aligned} V(t_0, x(\cdot)) \ge V(t, x(\cdot)) &\ge |x(t)| + K \int_0^t |C(t-s)|\,|x(s)|ds \\ &\ge [a(t)/a(t)]\left[|x(t)| + K \int_0^t |C(t-s)|\,|x(s)|ds\right] \\ &\ge M|x'(t)|/a(t), \quad \text{some } M > 0. \end{aligned}$$

The A.S. will follow from Theorem 9. This completes the proof.

REMARK. The growth of V, shown in (E5), is the property that allows us to place $|x'|$ in $\overline{V}'$ which, in turn, allows us to obtain the counterpart of Krasovskii's (ivb) in his first formulation.

The preceding examples all featured equations which were perturbations of U.A.S. ordinary differential equations. The next example shows that is not necessary. This is an interesting example in that it is not obvious by inspection that $V(t, x(\cdot)) \geq W(|x(t)|)$, yet the properties of V' show that this is true.

EXAMPLE F. Consider the scalar equation

$$x'(t) = \int_{-\infty}^{t} C(t-s)x(s)ds \tag{F1}$$

and suppose there is a function $G \in L^1[0, \infty)$ with

$$G'(u) = C(u), \quad \int_{t}^{\infty} |G(u)|du \geq K|C(t)|, \quad K > 0, \tag{F2}$$

and

$$2G(0) > [2G(0) + 1]\left[\int_{0}^{\infty} |G(u)|du\right] + 1. \tag{F3}$$

Then $x = 0$ is A.S. Moreover, if $k = G(0) + 1$ and

$$\begin{aligned} V(t, x(\cdot)) = \left[x - \int_{-\infty}^{t} G(t-s)x(s)ds\right]^2 \\ + k\int_{-\infty}^{t}\int_{t-s}^{\infty} |G(u)|du\, x^2(s)ds \end{aligned}$$

we have

$$V'(t, x(\cdot)) \leq -\alpha x^2, \quad \alpha > 0, \tag{F4}$$

$$V(t, x(\cdot)) \geq M|x'(t)|^2, \quad M > 0 \tag{F5}$$

so that

$$V'(t, x(\cdot)) \le -\big\{\alpha/[V(t_0, \phi)/M]^{1/2}\big\}x^2(t)|x'(t)| \tag{F6}$$

and

$$V(t_0, \phi) \ge \gamma\big[|\phi(t_0)|\big]^{1/2}, \quad \gamma > 0. \tag{F7}$$

PROOF. The equation may be expressed as

$$x'(t) = -G(0)x(t) + (d/dt)\int_{-\infty}^{t} G(t-s)x(s)ds \tag{F8}$$

and the details for (F4) are then found in Burton-Zhang [7]. We note that

$$|x'(t) \le \left[\int_{-\infty}^{t} |C(t-s)|ds \int_{-\infty}^{t} |C(t-s)|x^2(s)ds\right]^{1/2}$$

so that by (F2) we see that (F5) holds. Then $V' \le 0$ implies that $V(t_0,,\phi)/M \ge |x'(t)|^2$ so that (F6) then follows from (F4). The proof of Theorem 6 will show that if $t_0 \ge 0$ and $\phi : (-\infty, t_0] \to \mathbf{R}$ is any bounded and continuous function, then $x(t) = x(t, t_0, \phi) \to 0$ as $t \to \infty$. From (F4) and (F5) we see that

$$V^{1/2}(t, x(\cdot))V'(t, x(\cdot)) \le -\beta x^2(t)|x'(t)|, \quad \beta > 0$$

so that for $V(t) = V(t, x(\cdot))$ we have

$$\begin{aligned}(2/3)\big[V^{3/2}(t) - V^{3/2}(t_0)\big] &\le -\beta\int_{t_0}^{t} x^2(s)|x'(s)|ds \\ &\le -\beta\left|\int_{t_0}^{t} x^2(s)x'(s)ds\right| \\ &\le -(\beta/3)|x^3(t) - x^3(t_0)|.\end{aligned}$$

But $x(t) \to 0$ and $V(t) \to 0$ as $t \to \infty$ and so we obtain

$$(\beta/3)|x^3(t_0)| \le (2/3)V^{3/2}(t_0)$$

from which (F7) follows. This completes the proof.

In the next example it is not clear from inspection that $V(t, x_t) \geq W(\|x_t\|)$. Yet this can be obtained from $V(t, x_t) \geq |x(t)|$ and from $V' \leq -\alpha|x'(t)|$.

EXAMPLE G. Consider the scalar equation

$$x'(t) = -a(t)x(t) + \int_{t-h}^{t} b(s)x(s)ds \tag{G1}$$

with $a, b : [0, \infty) \to \mathbf{R}$ continuous,

$$0 < \theta_1 h < 1, \quad \theta > 0, \quad a(t) \notin L^1[0, \infty), \quad a(t) > 0. \tag{G2}$$

$$|b(t)| - \theta_1|a(t)| \leq -\theta_2|b(t)|. \tag{G3}$$

Then $x = 0$ is A.S. Also, if

$$V(t, x_t) = |x(t)| + \theta_1 \int_{-h}^{0} \int_{t+s}^{t} a(u)|x(u)|du\, ds$$

then

$$V'(t, x_t) \leq -\alpha\big[|x'(t)| + a(t)|x(t)|\big], \quad \alpha > 0. \tag{G4}$$

If $a(t) \geq a_0 > 0$, then for $t > t_0 + kh$ we have

$$V(t, x_t) \leq V(t_0, \phi) - \overline{\alpha} \sum_{j=1}^{k} \|x_{t_0+jh}\|, \quad \overline{\alpha} > 0. \tag{G5}$$

PROOF. A calculation yields

$$\begin{aligned}V'(t,x_t) &\le -a(t)|x(t)| + \int_{t-h}^{t} |b(s)|\,|x(s)|ds \\ &\quad + \theta_1 \int_{-h}^{0} \big[a(t)|x(t)| - a(t+s)|x(t+s)|\big]ds \\ &\le (\theta_1 h - 1)a(t)|x(t)| + \int_{t-h}^{t} \big[|b(s)| - \theta_1 a(s)\big]|x(s)|ds \\ &\le (\theta_1 h - 1)a(t)|x(t)| - \theta_2 \int_{t-h}^{t} |b(s)|\,|x(s)|ds\end{aligned}$$

so that (G4) holds. A.S. will now follow from Theorem 6. An integration of (G4) with $a(t) \ge a_0 > 0$ and application of Lemma 1 will prove (G5). This completes the proof.

EXAMPLE H. Hale [12; pp. 55–57] considers the scalar equation

$$x'(t) = -ax(t) - bx(t-h) \tag{H1}$$

with a and b constants and $h > 0$. He considers a functional

$$\begin{aligned}V(t,x_t) &= [x^2(t)/2|a|] + \alpha x(t) \int_{t-h}^{t} x(u)du \\ &\quad + \int_{t-h}^{t} \beta(u-t)x^2(u)du\end{aligned}$$

and, under complicated conditions, obtains

$$V'(t,x_t) \le -\gamma\bigg(x^2(t) + \int_{t-h}^{t} x^2(u)du\bigg). \tag{H2}$$

Note that

$$V'(t,x_t) \le -\gamma\bigg(x^2(t) + \int_{t}^{t+h} x^2(u-h)du\bigg)$$

so that if we define

$$\overline{V}(t,x(\cdot)) = V(t,x_t) + V(t-h,x_{t-h})$$

then there are positive constants γ_i with

$$\begin{aligned}\overline{V}'(t,x(\cdot)) &\le -\gamma_1\left[x^2(t)+x^2(t-h)+\int_{t-h}^{t}|x'(u)|^2du\right]\\ &\le -\gamma_2\left[|x(t)|+\int_{t-h}^{t}|x'(u)|^2|du\right]^2\\ &\text{(using Jensen's inequality)}\end{aligned}$$

so that

$$\overline{V}'(t,x(\cdot)) \le -\gamma_3\|x_t\|^2 \tag{H3}$$

by Lemma 1.

To this point we have looked at A.S. However, the ideas are fruitful for study of limit sets.

EXAMPLE 1. Krasovskii [15; pp. 173–174] considers the system

$$\begin{cases} x'(t)=y(t)\\ y'(t)=-Q(t,y(t))-f(x(t))+\displaystyle\int_{-h(t)}^{0} f^*(x(t+s))y(t+s)ds\end{cases} \tag{I1}$$

where

$$Q(t,y)/y \ge b > 0 \ \text{ for } \ y \ne 0, \quad 0 \le h(t) \le h \tag{I2}$$

$$f(x)/x > a > 0 \ \text{ for } \ x \ne 0, \tag{I3}$$

$$f^*(x)=(d/dx)f(x), \quad |f^*(x)| < N. \tag{I4}$$

He shows that if $h < a/N$ and if Q is periodic in t, then all solutions tend to zero. (Actually, it seems he needs to relate b to a, as well.) But if Q is unbounded in t, then it is known that even for $h(t) \equiv 0$, solutions need not tend to zero. We show that if

$$V(x_t, y_t) = y^2(t) + 2\int_0^x f(x)dx + N\int_{-h}^0\int_u^0 y^2(t+s)ds\,du$$

and if

$$-[Q(t,y)/y] + Nh \leq -\lambda Q(t,y)/y, \quad \lambda > 0, \tag{I5}$$

then

$$VV' \leq -\overline{\lambda}y^2(t)|y'(t)|, \quad \overline{\lambda} > 0 \tag{I6}$$

and $y(t) \to 0$ as $t \to \infty$ and $x(t) \to$ constant.

PROOF. We have

$$\begin{aligned} V'(x_t, y_t) &\leq -2Q(t,y)y + \int_{t-h}^t N[y^2(t) + y^2(s)]ds \\ &\qquad + N\int_{-h}^0 [y^2(t) - y^2(t+u)]du \\ &\leq \big[\{-2Q(t,y)/y\} + 2Nh\big]y^2 \\ &\leq -2\lambda Q(t,y)y. \end{aligned}$$

Using $f(x)/x > a$ we have

$$\int_0^x f(s)ds \geq \int_0^x as\,ds = ax^2/2.$$

Using $|f'(x)| < N$ we have

$$|f(x)| = \left|\int_0^x f'(s)ds\right| \leq \left|\int_0^x N\,ds\right| = N|x|.$$

These two relations yield

$$\int_0^x f(s)ds \geq (a/2N^2)f^2(x).$$

Moreover, if $y(t) \nrightarrow 0$, then there is a $\gamma > 0$ with $V(x_t, y_t) \geq \gamma$ and we have $V^{1/2} \geq \gamma^{1/2}$ so that $V \geq V^{1/2}\gamma^{1/2}$. Also, there is a $P > 0$ with $N \int_{t-h}^{t} y^2(u)du \leq P$ so $(N\gamma/2P) \int_{t-h}^{t} y^2(u)du \leq \gamma/2$. This means that $V = (V+V)/2 \geq (V/2) + (N\gamma/2P) \int_{t-h}^{t} y^2(u)du$ and so

$$
\begin{aligned}
&V(x_t, y_t)V'(x_t, y_t) \\
&\quad \leq -2\lambda Q(t,y)y\gamma^{1/2}V^{1/2} \\
&\quad \leq -2\lambda Q(t,y)y\gamma^{1/2}\left[(V/2) + (N\gamma/2P)\int_{t-h}^{t} y^2(u)du\right]^{1/2} \\
&\quad \leq -\lambda^* Q(t,y)y\left[y^2 + 2\int_0^x f(s)ds + N\int_{t-h}^{t} y^2(u)du\right]^{1/2} \\
&\quad \leq -\overline{\lambda} Q(t,y)y\left[|y| + \left[\int_0^x f(s)ds\right]^{1/2} + \int_{t-h}^{t} |f^*(x(s))y(s)|ds\right] \\
&\quad \leq -\overline{\overline{\lambda}} Q(t,y)y\left[|y| + |f(x)| + \int_{t-h}^{t} |f^*(x(s))y(s)|ds\right] \\
&\quad \leq -\tilde{\lambda} y^2\left[Q(t,y) + |f(x)| + \int_{t-h}^{t} |f^*(x(s))y(s)|ds\right] \\
&\quad \leq -\tilde{\lambda} y^2|y'|, \quad \tilde{\lambda} > 0.
\end{aligned}
$$

It now readily follows that $y(t) \to 0$.

Since $V' \leq 0$, $V(x_t, y_t) \to$ constant. But

$$V(x_t, y_t) \to 2\int_0^x f(s)ds$$

and so $x(t) \to$ constant. This completes the proof.

5 The general results and proofs

The following results were motivated by the examples of Section 4 and complete the proofs of those results. Theorem 6 is given mainly to complete the proof of Example A. Such results are discussed in Burton ([5; pp. 237–239 and [4]) for ordinary differential equations. It can be significantly improved when the delay is bounded by means of Jensen's inequality, particularly when $x = 0$ is U.S. Related results are found in ([14], [17], [20]).

THEOREM 6. Let $V(t, x(\cdot))$ be a continuous functional which is locally Lipschitz in $x(\cdot)$ for $|x(t)| < H$ and let $\eta : [0, \infty) \to [0, \infty)$ be continuous. If

(i) $W_1(|x(t)|) \leq V(t, x(\cdot))$, $V(t, 0) = 0$,

(ii) $V'_{(10)}(t, x(\cdot)) \leq -\eta(t) W_2(|x(t)|) - W_3(|x|)|x'(t)|$,

(iii) $\displaystyle\int_0^\infty \eta(t)dt = \infty$,

then $x = 0$ is A.S.

PROOF. The stability readily follows from (i) and (ii). Suppose there is a solution $x(t) = x(t, t_0, \phi)$ on $[t_0, \infty)$ with $|x(t)| < H$ which does not tend to zero. Then there is an $\epsilon > 0$ and $\{t_n\} \uparrow \infty$ with $|x(t_n)| > \epsilon$ on $[T, \infty)$. To show that this is impossible we first suppose there is a $T > t_0$ with $|x(t)| \geq \epsilon/2$ on $[T, \infty)$. Then $V'(t, x(\cdot)) \leq -\eta(t) W_2(\epsilon/2)$ which, by (iii) will contradict $V \geq 0$. Thus, for each n there is a T_n with $|x(t)| \geq \epsilon/2$ on $[t_n, T_n]$ and $|x(T_n)| = \epsilon/2$. We may suppose, by renaming if necessary, that $t_n < T_n < t_{n+1}$. Then from (ii) if $t > T_n$ we have

$$\begin{aligned} 0 \leq V(t, x(\cdot)) &\leq V(t_0, x(\cdot)) - \sum_{i=1}^{n} W_3(\epsilon/2) \int_{t_i}^{T_i} |x'(s)|ds \\ &\leq V(t, x(\cdot)) - W_3(\epsilon/2) n(\epsilon/2), \end{aligned}$$

a contradiction for large n. This completes the proof.

THEOREM 7. Suppose there is a continuous functional $V(t, x(\cdot))$ which is locally Lipschitz in $x(\cdot)$ for $|x(t)| < H$, a continuous function $g : [0, \infty) \to [0, \infty)$, an $\alpha > 0$, and a

$p > 1$ with

$$W_1|x(t)| \leq V(t, x(\cdot)), \quad V(t,0) = 0. \tag{i}$$

for each $t_0 \geq 0$, if t is large enough then

$$V(t_0, x(\cdot)) \geq \int_{g(t)}^{t} |x'(s)|^p ds, \tag{ii}$$

$$V'_{(10)}(t, x(\cdot)) \leq -W_2(|x(t)|), \tag{iii}$$

and

$$t - g(t) \geq \alpha \quad \text{for large } t. \tag{iv}$$

Then $x = 0$ is A.S.

PROOF. Stability follows from (i) and (iii). Suppose there is a solution $x(t) = x(t, t_0, \phi)$ of (10) with $|x(t)| < H$ and that $x(t) \nrightarrow 0$ as $t \to \infty$. There there is an $\epsilon > 0$ and $\{t_n\} \uparrow \infty$ such that $|x(t_n)| > \epsilon$. By (iii) we can argue that there is a sequence $\{T_n\}$ with $t_n < T_n + \alpha < t_{n+1}$ for which $|x(T_n)| = \epsilon/2$ and $|x(t)| \geq \epsilon/2$ on $[t_n, T_n]$, all of this for large n, say $n \geq 1$.

We prove the result for $p = 2$. First we note that there is no subsequence with $T_n - t_n \geq \alpha$ since that would yield $V'(t, x(\cdot)) \leq -W_2(\epsilon/2)$ on $[t_n, T_n]$, implying that $V(t, x(\cdot)) \to -\infty$. Hence, for large enough t we have from (ii) that $\int\limits_{g(t)}^{t} |x'(s)|^2 ds \leq V(t_0, x(\cdot))$ so that $\int\limits_{t_n}^{T_n} |x'(s)|^2 ds / V(t_0, x(\cdot)) \leq 1$. Then for $t > T_n$ we have

$$\begin{aligned} V(t, x(\cdot)) - V(t_0, x(\cdot)) &\leq -\sum_{j=1}^{n} \int_{t_n}^{T_n} W_2(|x(s)|) ds \\ &\leq -\sum_{j=1}^{n} \int_{t_n}^{T_n} W_2(|x(s)|) ds \int_{t_n}^{T_n} |x'(s)|^2 ds / V(t_0, x(\cdot)) \end{aligned}$$

$$\leq -\sum_{j=1}^{n}\left(\int_{t_n}^{T_n} W_2^{1/2}(|x(s)|)|x'(s)|^2 ds\right)^2 / V(t_0, x(\cdot))$$

$$\leq -[W_2(\epsilon/2)/V(t_0, x(\cdot))]\sum_{j=1}^{n}\left(\int_{t_n}^{T_n} |x'(s)|^2 ds\right)^2$$

$$\leq -W_2(\epsilon/2)(\epsilon/2)^2 n/V(t_0, x(\cdot)),$$

a contradiction to $V > 0$ for large n. This completes the proof.

LEMMA 4. Let W be a convex function from $[0,\infty)$ to $[0,\infty)$ with $W(0) = 0$. If $0 < a < 1$ and $b \geq 0$, then $W(b) \leq aW(b/a)$.

PROOF. We have

$$W(b) = W(ab/a) = W((1-a)0 + ab/a)$$

$$\leq (1-a)W(0) + aW(b/a) = aW(b/a).$$

This completes the proof.

THEOREM 8. Let $x = 0$ be U.S. for the finite delay system (2). Suppose that $V : [0,\infty) \times C_H \to [0,\infty)$ is continuous and that there is a continuous decreasing function $\eta : [0,\infty) \to [0,\infty)$, $\eta \notin L^1[0,\infty)$, with

(i) $0 \leq V(t, x_t)$,

(ii) $V'(t, x_t) \leq -W_1(|x(t)|)$,

(iii) $V(t, x_t) \geq W_2(|x'(t)|)W_3(|x(t)|)\eta(t)$, and

(iv) W_2 is convex.

Then $x = 0$ is A.S.

PROOF. Let $x(t) = x(t, t_0, \phi)$ be a solution of (2) with $|x(t)| < H$ and suppose $x(t) \not\to 0$ as $t \to \infty$. Then there is an $\epsilon > 0$ and a sequence $\{t_n\} \uparrow \infty$ with $|x(t_n)| > \epsilon$. Now for this $\epsilon > 0$ find the δ of U.S. On each interval of length h there is a t with $|x(t)| > \delta$ since

$|x(t_n)| > \epsilon$. Thus, we consider $I_j = [t_0+(j-1)h, t_0+jh]$ and find $s_j \in I_j$ with $|x(s_j)| > \delta$. By (ii) there is not an $\alpha > 0$ and a subsequence, say $\{s_n\}$ again, with $|x(t)| \geq \delta/2$ on $[s_n, s_n+\alpha]$. Hence, there is a sequence $\{T_n\}$ with $s_n < T_n < s_n + h$ and $|x(T_n)| = \delta/2$, $|x(t)| \geq \delta/2$ on $[s_n, T_n]$. Therefore, from (ii) and (iii) we have

$$V(t, x_t)V'(t, x_t) \leq -W_4(|x(t)|)W_2(|x'(t)|)\eta(t)$$

for $W_4(u) = W_3(u)W_1(u)$. If $t > t_0 + 2nh$ then

$$\begin{aligned}
V^2(t, x_t) - V^2(t_0, \phi) &\leq -2\sum_{j=1}^{2n} \int_{I_j} W_4(|x(s)|)W_2(|x'(s)|)\eta(s)ds \\
&\leq -2\sum_{j=1}^{2n} \eta(t_0 + jh) \int_{I_j} W_4(|x(s)|)W_2(|x'(s)|)ds \\
&\leq -2\sum_{j=1}^{n} \eta(t_0 + 2jh)W_4(\delta/2) \int_{s_{2j}}^{T_{2j}} W_2(|x'(s)|)ds \\
&\leq -2\sum_{j=1}^{n} \eta(t_0 + 2jh)W_4(\delta/2)h\big\{[T_{2j} - s_{2j}]/h\big\} \\
&\qquad \cdot W_2\Big[\int_{s_{2j}}^{T_{2j}} |x'(s)|ds/\big\{h[T_{2j} - s_{2j}]/h\big\}\Big] \\
&\qquad\qquad \text{(by Lemma 4)} \\
&\leq -2h\sum_{j=1}^{n} \eta(t_0 + 2jh)W_4(\delta/2)W_2[h\delta/2] \\
&\leq -2hW_4(\delta/2)W_2[\delta/2h] \sum_{j=1}^{n} \eta(t_0 + 2jh) \\
&\leq -2hW_4(\delta/2)W_2[\delta/2h] \sum_{j=1}^{n} \eta(t_0 + 2jh)
\end{aligned}$$

which tends to $-\infty$ as $n \to \infty$, a contradiction.

The following definition was introduced in [6]. It is a generalization of the common notion of integral positivity used in stability theory of ordinary differential equations.

DEFINITION. A measurable function $\eta : [0,\infty) \to [0,\infty)$ is said to be integrally positive with parameter $\delta > 0$ (IP(δ)) if whenever $\{t_i\}$ and $\{\delta_i\}$ satisfy $t_i + \delta_i < t_{i+1}$, $\delta_i \geq \delta$, then

$$\sum_{i=1}^{\infty} \int_{t_i}^{t_i+\delta_i} \eta(t)dt = \infty.$$

REMARK. If $\eta(t) \equiv 1$, then the U.S. in Theorem 8 may be deleted, as may be seen from the proof of the next result.

THEOREM 9. Consider the equation (10) and suppose $V(t, x(\cdot))$ is continuous and locally Lipschitz in $x(\cdot)$, $\eta \in$ IP(δ), $0 < \delta$, and that

(i) $W_1(|x(t)|) \leq V(t, x(\cdot))$, $V(t, 0) = 0$,

(ii) $V'_{(10)}(t, x(\cdot)) \leq -\eta(t)W_2(|x(t)|) - W_3(|x(t)|)W_4(|x'(t)|)$,

(iii) W_4 convex.

Then $x = 0$ is A.S.

PROOF. The stability follows from (i) and (ii). Suppose that $x(t) = x(t, t_0, \phi)$ is a solution with $V(t, x(\cdot))$ defined and that $x(t) \nrightarrow 0$ as $t \to \infty$. Then there is an $\epsilon > 0$ and a sequence $\{t_n\} \uparrow \infty$ with $|x(t_n)| \geq \epsilon$. By (ii) and the IP of η we see that there is not a subsequence, say $\{t_n\}$ again, with $|x(t)| \geq \epsilon/2$ on $[t_n, t_n + \delta]$. Thus, there is a sequence $\{T_n\}$ with $|x(T_n)| = \epsilon/2$, $|x(t)| \geq \epsilon/2$ on $[t_n, T_n]$, $T_n - t_n < \delta$, $T_n < t_{n+1}$ for large n, say $n \geq 1$. Thus, for $t > T_n$ we have

$$\begin{aligned}
V(t, x(\cdot)) - V(t_0, \phi) &\leq -\sum_{j=1}^{n} \int_{t_j}^{T_j} W_3(|x(s)|)W_4(|x'(s)|)ds \\
&\leq -W_3(\epsilon/2) \sum_{j=1}^{n} [T_j - t_j] W_4\Big(\int_{t_j}^{T_j} |x'(s)|ds/[T_j - t_j]\Big) \\
&\leq -W_3(\epsilon/2)\delta \sum_{j=1}^{n} \big([T_j - t_j]/\delta\big) W_4\big((1/\delta)[\epsilon/2]/([T_j - t_j]/\delta)\big)
\end{aligned}$$

$$\leq -W_3(\epsilon/2)\delta \sum_{j=1}^{n} W_4(\epsilon/2\delta) \to -\infty$$

as $n \to \infty$. This completes the proof.

THEOREM 10. Let $V(t, x(\cdot))$ be continuous and locally Lipschitz in $x(\cdot)$, $\eta_i : [0, \infty) \to [0, \infty)$ be continuous, $\eta_2(t) > 0$, $\eta_1 \in \mathrm{IP}(\delta)$ for some $\delta > 0$, $0 < p < 1$, $q = p/(p-1)$, with

$$W_1(|x(t)|) \leq V(t, x_t), \quad V(t, 0) = 0, \tag{i}$$

$$V'_{(10)}(t, x(\cdot)) \leq -\eta_1(t)W_3(|x(t)|) - \eta_2(t)W_4(|x(t)|)W_2(|x'|) \tag{ii}$$

where $[W_2(r)]^p$ is convex,

$$\int_{-\delta}^{0} [\eta_2(t+s)]^q ds < B, \quad 0 < B < \infty, \quad t \geq 0. \tag{iii}$$

Then $x = 0$ is A.S.

PROOF. From (i) and (ii) we deduce that $x = 0$ is stable.

Suppose there is a solution $x(t) = x(t, t_0, \phi)$ on $[t_0, \infty)$ with $V(t, x(\cdot))$ defined and that $x(t) \nrightarrow 0$ as $t \to \infty$. Then there is an $\epsilon > 0$ and a sequence $\{t_n\} \uparrow \infty$ with $|x(t_n)| \geq \epsilon$.

We first assume that there is a $T > t_0$ such that $|x(t)| > \epsilon/2$ for $t \geq T$. Then from (ii) we have

$$V(t, x(\cdot)) - V(t_0, \phi) \leq -\int_{t_0}^{t} \eta_1(s)W_3(|x(s|)ds$$

which tends to $-\infty$ as $t \to \infty$ because $\eta_1 \in \mathrm{IP}(\delta)$.

Thus, we may suppose that there exists $\{T_n\} \uparrow \infty$ such that $|x(t)| \geq \epsilon/2$ on $[t_n, T_n]$ and $|x(t_n)| = \epsilon/2$. We note that $T_n - t_n < \delta$ for all large n, say $n \geq 1$; for, if not, then we obtain a contradiction to $V \geq 0$ just as before.

Define $I_j = [t_j, t_j + \delta]$ and renumber, if necessary, so that the I_j are disjoint. Then for $t > t_n + \delta$ we have

$$\begin{aligned}
&V(t,x(\cdot)) - V(t_0, \phi) \\
&\quad \le -\int_{t_0}^{t} \eta_2(s) W_4(|(s)|) W_2(|x'(s)|) ds \\
&\quad \le -\sum_{j=1}^{n} \int_{I_j} \eta_2(s) W_4(|x(s)|) W_2(|x'(s)|) ds \\
&\quad \le -\sum_{j=1}^{n} \left\{ \int_{I_j} \big[W_4(|x(s)|) W_2(|x'(s)|)\big]^p ds \right\}^{1/p} \left\{ \int_{I_j} [\eta_2(s)]^q ds \right\}^{1/q} \\
&\quad \le -\sum_{j=1}^{n} \left\{ W_4^p(\epsilon/2) \int_{t_j}^{T_j} W_2(|x'(s)|)^p ds \right\}^{1/p} B^{1/q}
\end{aligned}$$

because

$$\int_{I_j} [\eta_2(s)]^q ds < B \quad \text{implies} \quad \left(\int_{I_j} [\eta_2(s)]^q ds \right)^{1/q} > B^{1/q}$$

since $q < 0$. This yields

$$V(t, x(\cdot)) - V(t_0, \phi) \le \sum_{j=1}^{n} B^{1/q} W_4(\epsilon/2) \left\{ \int_{t_j}^{T_j} \big[W_2(|x'(s)|)\big]^p ds \right\}^{1/p}.$$

If $[W_2(r)]^p = W_5(r)$, then Jensen's inequality yields

$$\begin{aligned}
\int_{t_j}^{T_j} W_5(|x'(s)|) ds &\ge [T_j - t_j] W_5\Big(\int_{t_j}^{T_j} |x'(s)| ds / [T_j - t_j] \Big) \\
&\ge \big([T_j - t_j]/\delta\big) \delta W_5\bigg([1/\delta] \int_{t_j}^{T_j} |x'(s)| ds / \big([T_j - t_j]/\delta\big) \bigg) \\
&\ge \delta W_5\big([1/\delta]\epsilon/2\big).
\end{aligned}$$

Hence

$$V(t, x(\cdot)) - V(t_0, \phi) \le -\sum_{j=1}^{n} B^{1/q} W_4(\epsilon/2) \{W_5(\epsilon/2\delta)\delta\}^{1/p}$$

which tends to $-\infty$ as $n \to \infty$. This completes the proof.

THEOREM 11. Let $\{Q_j\} = \{[t_j, T_j]\}$ with $T_j < t_{j+1}$ and let $\eta : [0,\infty) \to [0,\infty)$ be continuous. Suppose that $0 < p < 1$ and $q = p/(p-1)$. If there is a continuous $V(t, x(\cdot))$ which is locally Lipschitz in $x(\cdot)$ with

(i) $W_1(|x(t)|) \le V(t, x(\cdot))$, $V(t,0) = 0$,

(ii) $V'_{(10)}(t, x(\cdot)) \le -\eta(t) W_2(|x(t)|)$,

(iii) $0 \le \int\limits_{Q_j} \eta^q(s)ds \le M$, $0 < \int\limits_{Q_j} \eta(s)ds < 1$, $M > 0$,

then $\int\limits_{Q_j} |x(s)|ds \to 0$ as $t \to \infty$ for any solution $x(t)$ of (10) with $V(t, x(\cdot))$ defined on $[t_0, \infty)$.

PROOF. Stability follows from (i) and (ii). Hence, such solutions exist. In particular, let $|x(t)| < 1$ on $[t_0, \infty)$. We may assume W_2 convex. If $t > T_k$ then

$$\begin{aligned}
V(t, x(\cdot)) - V(t_0, \phi) &\le -\sum_{j=1}^{k} \int_{Q_j} \eta(s) W_4(|x(s)|)ds \\
&\le -\sum_{j=1}^{k} \int_{Q_j} \eta(s)ds\, W_2\left(\int_{Q_j} \eta(s)|x(s)|ds \Big/ \int_{Q_j} \eta(s)ds \right) \\
&\qquad\qquad\qquad\qquad \text{(by Jensen's inequality)} \\
&\le -\sum_{j=1}^{k} W_2\left(\int_{Q_j} \eta(s)|x(s)|ds \right) \quad \text{(by Lemma 4)} \\
&\le -\sum_{j=1}^{k} W_2\left[\left(\int_{Q_j} |x(s)|^p ds \right)^{1/p} \left(\int_{Q_j} \eta^q(s)ds \right)^{1/q} \right] \\
&\le -\sum_{j=1}^{k} W_2\left[\left(\int_{Q_j} |x(s)|ds \right)^{1/p} M^{-1/p} \right]
\end{aligned}$$

(because $|x(t)| < 1$). Hence, $\int\limits_{Q_j} |x(s)|ds \to 0$ as $j \to \infty$. This completes the proof.

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