A UNIFICATION THEORY OF KRASNOSELSKII FOR DIFFERENTIAL EQUATIONS

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Abstract. The theory of differential equations is very broad and contains many seemingly unrelated types of problems with markedly different methods of solution. It is very difficult to discern any unity in the theory. Yet, sixty years ago one of the foremost investigators, Krasnoselskii, suggested the possibility of finding unity. He claimed that the inversion of a perturbed differential operator yields the sum of a contraction and compact map. Accordingly, he proved a general fixed point theorem to cover this situation. In this paper we begin a long study with a view to putting his idea to the test. We begin with fractional differential equations of Caputo type, continue to neutral functional differential equations, and conclude with a study of an old problem of Volterra which continues to describe many important real-world problems. For these problems there is the perfect unity predicted by Krasnoselskii. It is an invitation to continue the study by examining other important real-world problems. SEE NOTE ON P. 18.

1. Introduction

To successfully apply fixed point theory to a specific problem there must be two equal partners: A fixed point theorem and a fixed point mapping. Sixty years ago Krasnoselskii [22] (see also [34, p. 31]) studied an old paper of Schauder [33] on elliptic partial differential equations and formulated a working hypothesis which we formalize as follows.

Krasnoselskii's Hypothesis: *The inversion of a perturbed differential operator yields the sum of a contraction and a compact map.*

To dispel the notion that this was an idea concerning an isolated example, note that he then obtained the following very general result.

Theorem 1.1 (Krasnoselskii)**.** *Let $(\mathcal{S}, \|\cdot\|)$ be a Banach space, M a closed, convex, nonempty subset of $\mathcal{S}$. Suppose that $A, B : M \to \mathcal{S}$ such that*

(i) $$x, y \in M \Rightarrow Ax + By \in M,$$

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A is continuous and

(ii) *AM resides in a compact set,*

(iii) *B is a contraction*

with constant $\alpha < 1$.
Then $\exists\, y \in M$ with $Ay + By = y$.

The theorem has proved to be important with many applications and generalizations. A paper by Park [31] is a readily accessible summary of papers on the subject, although there have been many subsequent contributions. See for example [1], [16], [17], [18]. It is a difficult theorem to apply, particularly because of (i) (see [5]) and also establishing that A is a compact operator when we seek solutions on an entire half-line, say $[0, \infty)$.

In a totally different vein, the hypothesis and the theorem suggest that very different differential equations can be inverted so as to define a map suited to this single fixed point theorem. This would provide a unification of a very diverse area of mathematics. We put the hypothesis to the test in this paper by inverting four perturbed differential operators. The first is a general scalar fractional differential equation of Caputo type. We had studied this problem in a series of papers and had constantly struggled with the compactness question. Finally, we resolved that question by going to a weighted space. It was that step which led to this study of unification. The first step was taken in [8] in which it was noted that fractional equations utilized only the compact map so that we were dealing with Schauder's theorem, not Krasnoselskii's. It was then noted that the compactness of the map in Schauder's theorem was not needed so that the resulting theorem for fractional equations was very close to the much simpler Brouwer's fixed point theorem. In the case of fractional differential equations we often could get by with a contraction mapping, but a compact map always does suffice.

The second step in our quest for a unified theory concerns neutral functional differential equations and this is divided into two parts depending on whether we deal with a stable or unstable problem. It is the unstable problem which has generated so much interest in problems in mathematical biology. The stable problem is inverted exactly as in the former studies. Because of the neutral term we need one integration by parts and that, for the first time, does necessarily yield the sum of two operators exactly as Krasnoselskii had promised. Again, the compactness of the operator can be completely ignored and our alternate theorem is applied directly.

This inversion is an interesting addition to the Krasnoselskii theory. Many investigators [18], [32] have applied his theorem directly to an integral equation having the sum of a Volterra and Hammerstein integrals, yielding the sum of a contraction and compact map on a bounded interval $[a, b]$. But this example inverts a problem yielding that sum and the compactness is on $[0, \infty)$.

The third step involves the classical unstable neutral functional differential equation used as a general logistic equation. We had studied it long ago and the inversion given there fits precisely into the sum of operators. Again, the compactness can be ignored and we use the alternate Krasnoselskii theorem to obtain a bounded solution on the entire half-line. It is a very sophisticated problem, but its solution turns out to be parallel to the solution of the classical elementary logistic equation $x' = ax - bx^2$ with a and b positive constants. The unstable part, $x' = ax$, is countered by the $-x^2$ to produce the bounded solution.

For our fourth step we consider a problem of Volterra

$$x'(t) = -\int_0^t D(t-s)g(x(s))ds,$$

which has many modern and classical applications to be mentioned later. The inversion and transformation work perfectly and we are left with a problem completely parallel to that studied by fixed point theory for fractional equations. Again, we need only Schauder's theorem and, again, the compactness of the mapping need never be mentioned. It is challenging to find the set M which is mapped into itself and a main contribution of this paper is to show exactly how to do that. We arrive at a new result which does not require the draconian conditions on $D(t)$ which have been required since the initial paper of Levin [28] in 1963. This is a model for scalar integrodifferential equations.

In summary, we have studied four unrelated fundamental problems and followed Krasnoselskii's Hypothesis to reduce them to exactly the same type of fixed point problem. These problems involve partial differential equations, integral equations, ordinary differential equations, Volterra integrodifferential equations, and neutral functional differential equations. It is just the beginning, but the idea is most intriguing that Krasnoselskii's Hypothesis could bring some unity to such a diverse area of mathematics.

We work in the space $(BC, \|\cdot\|)$ of bounded continuous functions $\phi : [h, \infty) \to \Re$ and always ask that the functions map bounded sets into bounded sets. We seek bounded and continuous solutions on the entire half-line. The inversion yields the sum of two operators

$$(P\phi)(t) = (B\phi)(t) + (A\phi)(t),$$

where B is a contraction, and A is not compact in that space. However, A does map a closed, bounded, convex set M into an equicontinuous set. If we were working in a space of continuous functions on a closed interval $[a, b]$, then A would be compact on M and Krasnoselskii's theorem would immediately apply. But our interest is in functions on an entire half-line. The problem is then transformed to a Banach space with weighted norm $(X, |\cdot|_g)$ and it is shown that B is still a contraction in that space, that A is continuous on M in that space, and that A is compact on M in that space. Hence, Krasnoselskii's theorem holds in that space and so there is a fixed point. This means that **for these four problems considered in this paper as they are inverted here** the following theorem is true.

Theorem 1.2 (Krasnoselskii Inversions)**.** *Let $(BC, \|\cdot\|)$ be the Banach space of bounded continuous functions $\phi : [0,\infty) \to \Re$, M a closed ball in BC of the form $M = \{\phi \in BC | a \leq \phi(t) \leq b, a < b\}$. Suppose that A, $B : M \to BC$ such that $P = A + B$ is* ***the inversion of any of the four aforementioned perturbed differential operators****. Assume that*

(i) $$x, y \in M \Rightarrow Ax + By \in M,$$

(ii) $$A \textbf{ is continuous},$$

(iii) $$B \text{ is a contraction}$$

with constant $\alpha < 1$.
Then $\exists\, y \in M$ with $Ay + By = y$.

Notice that compactness is never mentioned. Moreover, in both the case of the Caputo fractional differential equation and the classical Volterra problem the operator B is absent and, hence, so is the difficult condition (i). It is a very user friendly result. And that is the main object of this paper.

2. The Building Blocks

We begin by studying fractional differential equations which represent a very broad range of ordinary, partial, and integral equations. The work below was first detailed in [7]. Much background for the inversion is found in [14], [26], [27], while the subsequent monotonicity work and variation of parameters is found in [30] with specific page numbers in [7]. We started with the scalar equation of Caputo type

(1) $$^cD^q x(t) = u(t, x(t)), \quad 0 < q < 1, \quad x(0) \in \Re$$

with $u(t, x)$ continuous so that it could be inverted as

(2) $$x(t) = x(0) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} u(s, x(s)) ds$$

which obviously did not meet the conditions of Krasnoselskii's Hypothesis. But the kernel is completely monotone and there is a completely monotone resolvent kernel, $R(t)$, so that it can be transformed by a nonlinear variation of parameters formula to

$$x(t) = x(0)\left[1 - \int_0^t R(s)ds\right] + \int_0^t R(t-s)\left[x(s) + \frac{u(s,x(s))}{J}\right]ds \tag{3}$$

where J is an arbitrary positive constant and

$$\int_0^\infty R(s)ds = 1, \quad 0 < R(t) \le \frac{J}{\Gamma(q)}t^{q-1}. \tag{4}$$

Here,

$$z(t) = x(0)\left[1 - \int_0^t R(s)ds\right]$$

is the solution of a linear equation used in the transformation. It will relate to work with (8) later.

In a series of papers ([10], [11], [12], [13], [8]) we showed that (3) is very fixed point friendly. The strategy is to make all four classes of problems fit exactly a suitable form of (3), keeping in mind that a contraction term will be added. We obtain a mapping equation for each of these problems which we use in our Krasnoselskii type fixed point theorem.

There are four fundamental results which hold for all the problems considered here making the conditions of Krasnoselskii's theorem so much easier to apply. The first is taken from [10].

Let $(BC, \|\cdot\|)$ be the Banach space of bounded continuous functions on $[0,\infty) \to \Re$ with the supremum norm and let M be a closed, bounded, convex, nonempty subset of BC. From (3) we define a mapping by $\phi \in M$ implies that

$$(A\phi)(t) = x(0)\left[1 - \int_0^t R(s)ds\right] + \int_0^t R(t-s)\left[\phi(s) + \frac{u(s,\phi(s))}{J}\right]ds. \tag{5}$$

Theorem 2.1. *Suppose there is a constant $S > 0$ so that $x \in M$ and $0 \le t < \infty$ imply that $|u(t,x)| \le S$. Then there is a constant $D > 0$ so that if $x \in M$, $0 < q < 1$, $0 < t_2 < t_1 < \infty$ then*

$$\begin{aligned} L := \bigg| \int_0^{t_1} & R(t_1 - s)u(s,x(s))ds \\ & - \int_0^{t_2} R(t_2 - s)u(s,x(s))ds \bigg| \le D|t_1 - t_2|^q. \end{aligned}$$

This says that AM is equicontinuous. Thus, if we were working on an interval $[a,b]$ then A would be compact on M. But for the infinite interval A is not compact on M unless we ask draconian conditions on $u(t,x)$. If we maintain M as stated in BC, but move all the work to Banach space with weighted norm, then A will be compact on M. In that space, if A is continuous on M in the supremum norm, it is still continuous in the weighted norm. It is certainly valid to suggest that we by-pass BC and work directly with the weighted space. There are two excellent reasons for not doing so, and the reasons are related.

First, in the weighted space common words do not have common meaning and we are easily led astray. Bounded sets are not necessarily bounded in any of our common perceptions. We think of E^n in which sets are compact if they are closed and bounded. In the weighted space some of the compact sets which are most useful in applied mathematics are unbounded.

Next, we want our theorems to be stated in the universally familiar space $(BC, \|\cdot\|)$ so that the applied scientist uses the result directly without being challenged by the unusual properties of the weighted space.

We see that for this class of fractional differential equations, the inversion and transformation yields a compact map. There is no contraction involved. But in later problems we will have the inversion yielding the sum as in Krasnoselskii's theorem. We will then need the result that if B is a contraction in the supremum norm, it is still a contraction with the same constant in the weighted norm.

For our work here, the weight function, g, is taken as a completely arbitrary continuous and strictly increasing scalar function $g : [0,\infty) \to [1,\infty)$ such that $g \uparrow +\infty$. Then $(X, |\cdot|_g)$ is the Banach space of continuous functions $\phi : [0,\infty) \to \Re$ for which

$$|\phi|_g := \sup_{t\geq 0} \frac{|\phi(t)|}{g(t)} < \infty.$$

See [6, pp. 169-170] for properties of this space.

Here are the specific theorems. They are found in [13].

Theorem 2.2. *Let $K > 0$ be given and let $M \subset X$ be the closed set in $(X, |\cdot|_g)$ of continuous functions $\phi : [0,\infty) \to \Re$ with $\|\phi\| \leq K$, $M = \{\phi \in X | \|\phi\| \leq K\}$. If $A : M \to BC$ and is bounded in the supremum norm, say $\|AM\| \leq K^*$, then A is continuous on M in the norm $|\cdot|_g$.*

Theorem 2.3. *Let M be a closed ball in BC and let AM be bounded in BC. The set AM is contained in a compact subset of $(X, |\cdot|_g)$.*

The main parts of the next result were obtained in [13].

Theorem 2.4. *If for $x \in M$ and $t \geq 0$ the function $u(t,x)$ is continuous and satisfies a contraction condition for $\alpha < 1$ in the supremum norm, then $\int_0^t R(t-s)u(s,x(s))ds$ satisfies a contraction condition in both $\|\cdot\|$ and $|\cdot|_g$ for the same constant α, while $u(t,x)$ satisfies a contraction condition in $|\cdot|_g$ for the same α.*

Techniques for showing that $A : M \to M$ may be found in [10], [11], [12], [13], as well as Liapunov techniques in [2] and [9].

These building blocks tell us that for our fractional differential equation, we can state Krasnoselskii's theorem in $(BC, \|\cdot\|)$ asking only that A be continuous on M. Then, retaining this same M we change to $(X, |\cdot|_g)$ and find that A has become compact on M, that A retains its continuity, and that B has retained its contraction property. **Compactness is never mentioned in the result and the investigator need never consider the question.** Krasnoselskii's theorem yields a fixed point in M. Our task now is to enlarge the set of problems for which this simplification is still true, beyond the set of fractional differential equations.

The following result was obtained in [8] and it shows that when B is absent so is the difficult condition (i) (see, [5]) of Krasnoselskii's theorem and we are left with Schauder's theorem. But we can ignore the requirement of compactness of the operator A and obtain a theorem which is very much like Brouwer's much simpler result for E^n.

Theorem 2.5 (Brouwer-Schauder). *Let M be a closed ball in $(BC, \|\cdot\|)$. Suppose that $x(0)$ and M are chosen so that for A defined in (5) then $A : M \to M$. If A is continuous and if there is an $L > 0$ so that $|u(t, x)| \leq L$ for $x \in M$, then A has a fixed point in M.*

We will apply the same theorem when we deal with our last problem involving Volterra's equation.

The mapping A in (5) has a very interesting property relative to the statement of Theorem 2.5. Suppose for that theorem we have found M to be the set of $x \in BC$ with $\|x\| \leq D$ so that for $x \in M$ we have

$$\int_0^t R(t-s)\left|x(s) + \frac{u(s, x(s))}{J}\right| ds \leq \int_0^t R(t-s) D ds.$$

Now, in that theorem let us choose $|x(0)| \leq D$. Notice that for $x \in M$ we then have

$$\begin{aligned} |(Ax)(t)| &\leq |x(0)| \left[1 - \int_0^t R(s)ds\right] + \int_0^t R(t-s) D ds \\ &\leq D\left[1 - \int_0^t R(s) + \int_0^t R(s)ds\right] = D. \end{aligned}$$

The initial condition is added without charge!

Neutral functional differential equations give rise to an extended form of (3), namely

$$x(t) = f(t) + \alpha x(t-h) + \int_0^t R(t-s)\Big[x(s) - u(s, x(s), x(s-h))\Big] ds.$$

This not only allows us to add another significant area of study to our group, but it is the first one which now does require the sum of a contraction and a compact map.

For these four areas of differential equations the desired unification has been achieved. The mappings for each set are so similar that all the foundation laid in [10], [11], [12], and [13] for fractional equations stands us in good stead for the other three areas.

All of the details in (2) through (5) proceed in exactly the same way for a scalar fractional differential equation with a delay,

$$^{c}D^{q}x(t) = u(t, x(t), x(t-r)), 0 < q < 1, 0 < r, x(t) = \psi(t), -r \le t \le 0.$$

3. Stable neutral functional differential equations

A differential equation of the form

$$x'(t) = f(t, x(t))$$

states that the growth or decay of $x(t)$ depends on its position and the time. In many living things, growth depends also on the rate of recent growth:

$$x'(t) = F(t, x(t), x(t-h), x'(t-h))$$

where h is a positive constant. We later give a number of fundamental references. But here is a simple heuristic explanation. It is so common that all parents of growing children observe it. Typically, a child begins to grow more rapidly at the age of about 12 years, growing more and more rapidly until a certain height is approached, at which time there is a rapid slowing of growth, stopping at the adult height dictated by genes. The growth rate is depending on recent growth rate, as well as the position and time. This process is easily interpreted in that last equation.

We consider a neutral functional differential equation

$$x'(t) = \alpha x'(t-h) - u(t, x(t), x(t-h)), \quad x(t) = \psi(t), -h \le t \le 0. \tag{6}$$

Here, h is a positive constant, u is continuous, $|\alpha| < 1$, and ψ is the initial function with $x(0) = \psi(0)$. It is critical here that ψ have a continuous derivative as that relates to continuity and differentiability of the solution, a necessary condition in an integration by parts which is to follow. There is a large and well-developed theory of neutral equations with a starting point in [15] and [21, pp. 24-35] where the main contribution is an excellent set of references.

If we denote the left-hand-derivative of ψ at zero by $\psi'(0)$ and the right-hand-derivative at $-h$ by $\psi'(-h)$ then a suitable condition is

$$\psi'(0) = \alpha\psi'(-h) - u(0, \psi(0), \psi(-h)) \tag{7}$$

and that will give $x(t)$ in (6) a continuous derivative. Continuity of the derivative is discussed in [21, p. 25].

The reader will follow the technical details of transforming (6) into the mapping without reference to the physical problem or the conditions for differentiability. The constant α will satisfy $|\alpha| < 1$.

The work proceeds exactly as it did for fractional equations and we will shortly arrive at corresponding functions ($J > 0$ is constant)

$$z(t) = \psi(0)e^{-Jt} \quad \text{and} \quad R(t) = Je^{-Jt}$$

so that R is again completely monotone, $z(t) \to 0$, and $\int_0^\infty R(t)dt = 1$.

An integration of (6) yields

$$x(t) = \psi(0) + \int_0^t [\alpha x'(s-h) - u(s, x(s), x(s-h))]ds \tag{8}$$

with $x(t) = \psi(t)$ for $-h \le t \le 0$ so that everything in (8) is defined and a fixed point mapping could be deduced from it. That would not be very satisfactory because the integral would not map bounded sets into bounded sets, let alone into equicontinuous sets.

The simple linearization process is again utilized. In the integrand, multiply and divide by $J > 0$, subtract and add $x(s)$, separate out the linear part as

$$z(t) = \psi(0) - \int_0^t Jz(s)ds$$

yielding $z(t) = \psi(0)e^{-Jt}$. The result, easily checked by differentiation, is

$$x(t) = z(t) + \int_0^t Je^{-J(t-s)}\left[x(s) + \frac{\alpha x'(s-h)}{J} - \frac{u(s, x(s), x(s-h))}{J}\right]ds.$$

An integration of a term from the integrand by parts yields

$$\int_0^t e^{-J(t-s)}\alpha x'(s-h)ds = \alpha x(t-h) - \alpha e^{-Jt}x(-h) - \int_0^t J\alpha e^{-J(t-s)}x(s-h)ds.$$

This leads us to

$$\begin{aligned} x(t) &= z(t) + \alpha x(t-h) - \alpha e^{-Jt}x(-h) \\ &+ \int_0^t Je^{-J(t-s)}\left[-\alpha x(s-h) + x(s) - \frac{u(s, x(s), x(s-h))}{J}\right]ds. \end{aligned} \tag{9}$$

We have now come to a very instructive example of Krasnoselskii's hypothesis and his theorem. First, we take $(BC, \|\cdot\|)$ to be the Banach space of bounded continuous functions on $[-h, \infty) \to \Re$. Then we take M to be a closed, bounded, convex subset of BC such that if $\phi \in M$ then $\phi(t) = \psi(t)$ for $-h \le t \le 0$. Next, we use (9) to define a mapping of $M \to BC$ by $\phi \in M$ implies that $(P\phi)(t) = \psi(t)$ for $-h \le t \le 0$ and for $t > 0$ then

$$\begin{aligned} (P\phi)(t) &= z(t) + \alpha\phi(t-h) - \alpha e^{-Jt}\psi(-h) \\ &\quad + \int_0^t Je^{-J(t-s)}\left[-\alpha\phi(s-h) + \phi(s) - \frac{u(s, \phi(s), \phi(s-h))}{J}\right]ds \\ &=: (B\phi)(t) + (A\phi)(t) \end{aligned} \tag{10}$$

where B is the set of terms outside the integral, while A is the integral. Theorem 2.1 will give us equicontinuity of AM.

Notice that in (10) B does not smooth so if u does not define a contraction then we necessarily have the sum of a contraction and a compact map.

Theorem 3.1. *If there is a* $K > 0$ *such that for* $\phi \in M$ *and* $0 \le t < \infty$ *we have* $|u(t, \phi(t), \phi(t-h))| \le K$*, if* $|\alpha| < 1$*, then (10) is the sum of a contraction and an equicontinuous map.*

We can now state Theorem 1.2 for (10) as follows.

Theorem 3.2 (Brouwer-Krasnoselskii I)**.** *Let* $(BC, \|\cdot\|)$ *be the Banach space of bounded continuous functions* $\phi : [-h, \infty) \to \Re$*,* M *a closed ball in* BC*. Suppose that* A*,* $B : M \to BC$ *satisfy the conditions of Theorem 3.1 for (10). Assume that*

(i) $$x, y \in M \Rightarrow Ax + By \in M,$$

(ii) $$A \text{ is continuous},$$

(iii) $$B \text{ is a contraction}$$

with constant $\alpha < 1$.
Then $\exists\ y \in M$ *with* $Ay + By = y$.

Remark In the program we outlined, we found that inversion of our fractional differential equations generated a compact map. In this, our second step, we finally find that the inversion of neutral differential equations does yield the contraction and compact map after changing norms. Moreover, we do not see how the contraction could be avoided. One more long standing question can be answered. Item (i) of Krasnoselskii's theorem has been a constant problem [5]. We reason that it really should require $x \in M \implies Ax + Bx \in M$. But looking at our mapping we see $x(t-h)$, $x(s-h)$, and $x(s)$. In our computations which follow, we must view these as three separate functions, being distinguished only as having a certain magnitude. This suggests the necessity of (i).

The integration in (8) and the transformation to (10) were based on the tacit assumption that $u(t, x(t), x(t-h))$ has the sign of $x(t)$ so that the equation is stable, as indicated in the last sentence of this section. Arriving at (10) was nontrivial and arriving at a parallel mapping equation in the unstable case is far more nontrivial. We will shortly consider a "logistic" equation

$$x'(t) = \alpha x'(t-h) + ax(t) - q(t, x(t), x(t-h))$$

with $a > 0$, a perturbation of the unstable equation $x' = ax$. A totally different mapping equation will be derived; yet, it will be of the Krasnoselskii form of the sum of a contraction and compact map. It will have properties equivalent in a general sense to those of the form (5), but the integral is from t to ∞; nevertheless, the basic properties are exactly the same.

Example. We will now specify the relations to ensure that there is a suitable set M with the conditions of Theorem 3.2 satisfied. Let

$M \subset BC$ be defined by $K > 0$ such that

$$M = \{\phi : [-h, \infty) \to \Re | \phi(t) = \psi(t) \text{ on } [-h, 0] \text{ and } \|\phi\| \le K\}.$$

Assume that there are $J > 0$, $\beta > 0$, $\gamma > 0$ so that

$$\begin{aligned} & 2|\alpha| + \max\{\beta, \gamma\} \le 1 \text{ and} \\ (11) \qquad & |x|, |y| \le K \implies \left| x - \frac{u(t, x, y)}{J} \right| \le \beta K. \end{aligned}$$

If $|\psi(0)| \le \gamma K$ then $x, y \in M$ implies $Ax + By \in M$. Moreover, if $(X, |\cdot|_g)$ is defined just before Theorem 2.2, then B is a contraction in the g-norm and A is a continuous map of $M \to M$ in the g-norm. Also, AM resides in a compact subset in the $g-$norm.

We now show that $x, y \in M$ implies $Ax + By$ in M. The set M consists of functions whose supremum norm is bounded by K so take absolute values throughout (10), replace $|\psi(0)|$ by γK, replace all the other functions by K, with

$$\left| x - \frac{u(s, x(s), x(s-h))}{J} \right| \le \beta K.$$

For the right-hand-side of (9) have

$$\begin{aligned} & (\gamma K + |\alpha| K) e^{-Jt} + |\alpha| K + (|\alpha| K + \beta K) \int_0^t J e^{-Js} ds \\ = \; & K \left[(\gamma + |\alpha|) e^{-Jt} + |\alpha| + (|\alpha| + \beta) \left(1 - e^{-Jt}\right) \right] \\ = \; & K \left\{ [(\gamma + |\alpha|) - (|\alpha| + \beta)] e^{-Jt} + |\alpha| + |\alpha| + \beta \right\} \\ = \; & K \left[(\gamma - \beta) e^{-Jt} + 2|\alpha| + \beta \right]. \end{aligned}$$

If $\gamma - \beta \ge 0$ then

$$\begin{aligned} K \left[(\gamma - \beta) e^{-Jt} + 2|\alpha| + \beta \right] & \le K [(\gamma - \beta) + 2|\alpha| + \beta] \\ & = K (2|\alpha| + \gamma), \end{aligned}$$

while if $\gamma - \beta < 0$ then

$$\begin{aligned} K \left[(\gamma - \beta) e^{-Jt} + 2|\alpha| + \beta \right] & \le K [0 + 2|\alpha| + \beta] \\ & = K (2|\alpha| + \beta), \end{aligned}$$

and so

$$K [(\gamma + |\alpha|) - (|\alpha| + \beta)] e^{-Jt} + K (2|\alpha| + \beta) \le K (2|\alpha| + \max\{\beta, \gamma\}).$$

In view of (11) we have that the right-hand-side of (9) is bounded by K. We summarize the above, in the following result.

Assume that there is a $J > 0$ and a $K > 0$ such that

(*) $|x|, |y| \le K$, *implies* $|Jx - u(t, x, y)| \le (1 - 2|\alpha|) JK, \quad t \in \mathbb{R}$,

and

$$|\psi(0)| \le (1 - 2|\alpha|) K.$$

Then the conditions of Theorem 3.2 for the set M are satisfied.

Consequently, if for some $J > 0$ we have that (*) holds for arbitrarily large values of K, then the conditions of Theorem 3.2 will be satisfied regardless of the value of $|\psi(0)|$. We have the following:

If there exists a $J > 0$ such that

$$\liminf_{\rho\to\infty}\left[\frac{1}{\rho}\sup_{|x|,|y|\le\rho}|Jx - u(t,x,y)|\right] < (1-2|\alpha|)J,$$

then the conditions of Theorem 3.2 for the set M are satisfied.

Indeed, for the given $J > 0$ we can choose a sufficiently large $\rho > 0$ with

$$|\psi(0)| \le (1-2|\alpha|)\rho,$$

and

$$\sup_{|x|,|y|\le\rho}|Jx - u(t,x,y)| \le (1-2|\alpha|)J\rho,$$

i.e., (*) holds with $K = \rho$.

Now let us consider the case that $xu(t,x,y) \ge 0$, $x, y, t \in \mathbb{R}$ and assume that for some given $J > 0$ there exists a $K > 0$ with

$$2|\alpha|J|x| \le |u(t,x,y)| \le 2(1-|\alpha|)J|x|, \quad \text{for } |x|,|y| \le K,\ t\in\mathbb{R}.$$

It follows that for $|x|, |y| \le K$, $t \in \mathbb{R}$ we have

$$\begin{aligned} -2|\alpha|J|x| &\ge -|u(t,x,y)| \ge -2(1-|\alpha|)J|x|, \\ Jx - 2|\alpha|J|x| &\ge J|x| - |u(t,x,y)| \ge J|x| - 2(1-|\alpha|)J|x|, \\ (1-2|\alpha|)J|x| &\ge J|x| - |u(t,x,y)| \ge (-1+2|\alpha|)J|x|, \end{aligned}$$

and

$$|Jx - u(t,x,y)| = |J|x| - |u(t,x,y)|| \le (1-2|\alpha|)J|x| \le (1-2|\alpha|)JK$$

i.e., (*) is satisfied. We note that if $xu(t,x,y) \le 0$, $x, y, t \in \mathbb{R}$ then (*) is not satisfied as, for $|x| = K$, we have

$$(1-2|\alpha|)JK < JK = |Jx| \le |Jx - u(t,x,y)|.$$

4. Unstable neutral functional differential equations

Let $|\alpha| < 1$, $a > 0$, $h > 0$, q be continuous and consider the neutral functional differential equation

$$x' = \alpha x'(t-h) + ax - q(t,x,x(t-h)). \tag{12}$$

The general theory is well-developed and was discussed and general references cited in the previous section. Such equations have been considered in mathematical biology. See, for example, Gopalsamy [19], Gopalsamy and Zhang [20], Kuang [23], [24], [25]. Investigators have given heuristic arguments to support their use in describing biological phenomena and much of this is formalized in the final chapter of each of the books by Gopalsamy [19] and Kuang [23].

Following [4], write the equation as

$$[x - \alpha x(t-h)]' = a[x - \alpha x(t-h)] + a\alpha x(t-h) - q(t,x,x(t-h)),$$

multiply by e^{-at}, and group terms as

$$\big[(x - \alpha x(t-h))e^{-at}\big]' = \big[a\alpha x(t-h) - q(t, x, x(t-h))\big]e^{-at}.$$

Notice that if x is continuous, so is $(x - \alpha x(t-h))'$. We search for a solution having the property that

$$[x(t) - \alpha x(t-h)]e^{-at} \to 0 \text{ as } t \to \infty \tag{13}$$

so that an integration from t to infinity yields

$$-[x(t) - \alpha x(t-h)]e^{-at} = \int_t^\infty \big[a\alpha x(s-h) - q(s, x(s), x(s-h))\big]e^{-as}ds$$

and, finally,

$$x(t) = \alpha x(t-h) + \int_t^\infty \big[q(s, x(s), x(s-h)) - a\alpha x(s-h)\big]e^{a(t-s)}ds. \tag{14}$$

As $|\alpha| < 1$, the first term on the right defines a contraction. We have inverted (12) without the struggle to make x' continuous.

We will select a set M and from (14) we define our mapping $P : M \to M$ by $\phi \in M$ implies that $(P\phi)(t) = \psi(t)$ on $[-h, 0]$ and for $t > 0$ then

$$(P\phi)(t) = \alpha\phi(t-h) + \int_t^\infty [q(s, \phi(s), \phi(s-h)) - a\alpha\phi(s-h)]e^{a(t-s)}ds$$
$$=: (B\phi)(t) + (A\phi)(t). \tag{15}$$

Theorem 4.1. *Let M be a subset of $(BC, \|\cdot\|)$ with the property that $\phi \in M$ implies that $u(t, \phi(t), \phi(t-h))$ is continuous for $0 \le t < \infty$ and there is a $K > 0$ such that $|u(t, \phi(t), \phi(t-h))| \le K$ for $0 \le t < \infty$. Then*

$$AM := \left\{\xi \in BC | \xi(t) = \int_t^\infty e^{a(t-s)}u(s, \phi(s), \phi(s-h))ds, \phi \in M\right\}$$

is an equicontinuous set of functions on $[0, \infty)$.

Proof. Notice that if $0 \le t_1 < t_2$ then for $\xi \in AM$ we have

$$\begin{aligned}
|\xi(t_1) - \xi(t_2)| &\le \bigg|\int_{t_1}^\infty e^{a(t_1-s)}u(s, \phi(s), \phi(s-h))ds \\
&\quad - \int_{t_2}^\infty e^{a(t_2-s)}u(s, \phi(s), \phi(s-h))ds\bigg| \\
&\le \int_{t_1}^{t_2} e^{a(t_1-s)}|u(s, \phi(s), \phi(s-h))|ds \\
&\quad + \int_{t_2}^\infty \left|e^{a(t_1-s)} - e^{a(t_2-s)}\right||u(s, \phi(s-h)|ds \\
&\le K\int_{t_1}^{t_2} e^{a(t_1-s)}ds + K\int_{t_2}^\infty \left|e^{a(t_1-s)} - e^{a(t_2-s)}\right|ds.
\end{aligned}$$

It is now a routine calculation to show that this tends to zero as $|t_1 - t_2| \to 0$. □

With this result the Krasnoselskii theorem which applies to this problem is almost identical to the one for the stable case. We state it as follows.

Theorem 4.2 (Brouwer-Krasnoselskii II)**.** *Let $(BC, \|\cdot\|)$ be the Banach space of bounded continuous functions $\phi : [-h, \infty) \to \Re$, M a closed ball in BC. Suppose that A and B are defined in (15), that A, $B : M \to BC$ such that*

$$\text{(i)} \qquad x, y \in M \Rightarrow Ax + By \in M.$$

Let A be continuous and let

$$\text{(ii)} \qquad B \text{ be a contraction}$$

with constant $\alpha < 1$.
Then $\exists\ y \in M$ with $Ay + By = y$.

Theorem 4.3. *Suppose there is a $K > 0$ such that $x \in BC, \|x\| \leq K$ implies that*

$$\left|\frac{1}{a}q(t, x(s), x(s-h)) - \alpha x(s-h)\right| \leq (1 - |\alpha|)K.$$

Let

$$M = \{\phi \in BC | \phi(t) = \psi(t), -h \leq t \leq 0, \|\phi\| \leq K\}$$

and define P as in (15). Then there is a $\phi \in M$ such that $P\phi = \phi$, a solution of (14). As $\phi \in M$, condition (13) holds.

Clearly, $P : M \to M$, while PM is equicontinuous and resides in a compact subset of $(X, |\cdot|_g)$.

5. Equations with an integral delay

We come now to a very important classical and modern equation with many real-world applications. A complete presentation including nonlinear aspects and forcing functions is very lengthy and will be offered elsewhere. For the present we will consider a linear problem and offer a simple solution, all within the general framework of Krasnoselskii's Hypothesis. This brief version is offered as further evidence of the validity of Krasnoselskii's unification.

In 1928 Volterra [35] noted that a number of real-world problems were being modelled by an integrodifferential equation

$$x'(t) = -\int_0^t D(t-s)g(x(s))ds. \tag{16}$$

He suggested that a Liapunov functional might be found and Levin [28] did find one in 1963 under the assumptions that

$$D(t) > 0, \quad D'(t) \leq 0, \quad D''(t) \geq 0 \tag{17}$$

and

$$xg(x) > 0 \text{ if } x \neq 0, \tag{18}$$

g is continuous. Levin and co-authors extended that work to more general problems, but always under assumptions parallel to (17). The same method required that

$$\int_0^x g(s)ds \to \infty \text{ as } |x| \to \infty \; . \tag{19}$$

The process continued and we see viscoelasticity [21, p. 120], circulating fuel nuclear reactors [29], neural networks [3], and many other modern problems modelled in this way. In some of the problems of this type the integral was truncated and the equation became

$$x'(t) = -\int_{t-L}^{t} D(t-s)g(x(s))ds$$

where $D(t) = 0$ if $t \geq L$.

Our work here is motivated in two very different ways. First, no one can seriously claim that real-world problems with all their uncertainties can be modeled with such exactitude as (17). We will ask instead that D be continuous,

$$D(t) > 0, \quad \int_0^\infty D(t)dt < \infty \tag{20}$$

and for our brief presentation here that

$$g(x) = x. \tag{21}$$

Finally, the wide application already seen suggests that even more application of far greater diversity may be possible in the way of fractional differential equations of Caputo type

$${}^cD^q x(t) = -\int_0^t D(t-s)x(s)ds, \quad x(0) \in \Re, \quad 0 < q < 1.$$

We can write this as the integral equation

$$x(t) = x(0) - \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \int_0^s D(s-u)x(u)duds,$$

and take the formal limit as $q \uparrow 1$ and obtain

$$x(t) = x(0) - \int_0^t \int_0^s D(s-u)x(u)duds$$

which we would get by direct integration. While we do not do so here, we could allow a singularity in D at 0 so long as the integral exists.

The idea is to proceed in exactly the same way as we did for fractional differential equations which resulted in such a fixed point friendly form.

Integrate (16), divide and multiply by $J > 0$, add and subtract $x(s)$ to obtain

$$x(t) = x(0) - \int_0^t J\left[x(s) - x(s) + \frac{\int_0^s D(s-u)x(u)du}{J}\right]ds.$$

Write the linear part as

$$z(t) = x(0) - \int_0^t Jz(s)ds$$

so that there is a resolvent equation

$$R(t) = J - \int_0^t JR(s)ds$$

with solution

$$R(t) = Je^{-Jt}$$

which is completely monotone and satisfies

$$\int_0^\infty R(s)ds = 1.$$

We then have

$$z(t) = x(0)\left[1 - \int_0^t R(s)ds\right]$$

and by a variation of parameters formula

$$x(t) = z(t) + \int_0^t R(t-s)\left[x(s) - \frac{\int_0^s D(s-u)x(u)du}{J}\right]ds. \tag{22}$$

This is exactly what we have done in the earlier papers on fractional differential equations.

Now, we prepare the integrand in (22):

$$\begin{aligned}
&\int_0^t R(t-s)\int_0^s \frac{D(s-u)}{J}x(u)duds \\
&= \int_0^t \int_u^t R(t-s)\frac{D(s-u)}{J}x(u)dsdu \\
&= \int_0^t \int_u^t R(t-s)\frac{D(s-u)}{J}dsx(u)du.
\end{aligned}$$

Write (22) as

$$x(t) = z(t) + \int_0^t \left[R(t-u) - \int_u^t R(t-s)\frac{D(s-u)}{J}ds\right]x(u)du. \tag{23}$$

Theorem 5.1. *If*

$$|x(0)|\, e^{-Jt} + \int_0^t \left|R(t-u) - \int_u^t R(t-s)\frac{D(s-u)}{J}ds\right| du \le 1 \tag{24}$$

for $0 \le t < \infty$*, then the natural mapping defined by (23) of the set of all bounded continuous functions*

$$M = \{\phi : [0,\infty) \to \Re | \|\phi\| \le 1\}$$

maps $M \to M$ *and Theorem 2.5 will give a fixed point in* M*.*

Proof. The natural mapping of M into itself from (23) is

$$(P\phi)(t) = z(t) + \int_0^t \left[R(t-u) - \int_u^t R(t-s)\frac{D(s-u)}{J}ds \right] \phi(u)du.$$

With (24) holding it is easily verified that $\|P\phi\| \le 1$.

The existence of the fixed point is now exactly as in Theorem 2.5. □

Our problem is to carefully show that the inequality (24) holds. If we integrate each term, each is bounded by 1 under simple assumptions on D (go back and interchange the order of integration again on the second term). Remember that R is known and that we have complete freedom to choose J and we can place magnitude conditions on D.

For $R(t) = Je^{-Jt}$ then (24) is

(25)

$$|x(0)|\, e^{-Jt} + \int_0^t \left| Je^{-J(t-u)} - \int_u^t e^{-J(t-s)} D(s-u)ds \right| du \le 1, \quad t \ge 0.$$

The following lemma shows so clearly that the classical conditions (17) are in no respect necessary. The function D need not even be bounded, let alone be decreasing. Neither the first nor the second derivative need exist. Thus, it is showing us that the conditions in (17) can be significantly perturbed without destroying the conclusion. The conditions in (20) can reasonably be expected to be satisfied and verified in real-world problems.

Lemma 5.2. *Let* $J > 0$ *and suppose that* D *satisfies (20). If*

$$\int_0^\infty e^{Jv} D(v)\, dv \le J, \tag{26}$$

then (25) is always true.

Proof. Fix an arbitrary $t > 0$. For $0 \le u \le t$ we set

$$\begin{aligned} R(u;t) &= R(t-u) = Je^{-J(t-u)}, \\ T(u;t) &= \int_u^t R(t-s)\frac{D(s-u)}{J}ds = \frac{1}{J}\int_u^t Je^{-J(t-s)}D(s-u)\, ds \\ &= \int_u^t e^{-J(t-s)}D(s-u)\, ds = e^{-Jt}\int_u^t e^{Js}D(s-u)\, ds. \end{aligned}$$

Denote by $\mathcal{C}_R$ and $\mathcal{C}_T$ the graphs of R and T, respectively, on $[0,t]$. If the two graphs meet at some $u_0 \in (0,t)$, then

$$\begin{aligned} Je^{-J(t-u_0)} &= \int_{u_0}^{t} e^{-J(t-s)} D(s-u_0)\, ds, \\ J &= \int_{u_0}^{t} e^{J(t-u_0)} e^{-J(t-s)} D(s-u_0)\, ds = \int_{u_0}^{t} e^{J(s-u_0)} D(s-u_0)\, ds; \end{aligned}$$

that is,

$$J = \int_0^{t-u_0} e^{Jv} D(v)\, ds.$$

Because of the positivity of the integrand $e^{Jv}D(v)$ the integral $\int_0^u e^{Jv} D(v)\, ds$ is an increasing function in u, so, in view of (26) we have $\int_0^{t-u_0} e^{Jv} D(v)\, ds < J$ for any $u_0 \in (0,t)$ regardless of the fixed (but arbitrary) t. Thus $\mathcal{C}_R$ and $\mathcal{C}_T$ do not meet for any $t \geq 0$. As $T(t;t) = 0 < J = R(t;t)$, it follows that $T(u;t) \leq R(u;t)$ for $0 \leq u \leq t$, so the argument in the absolute value in (25) is nonnegative. Hence,

$$\begin{aligned} & |x(0)|\, e^{-Jt} + \int_0^t \left| Je^{-J(t-u)} - \int_u^t e^{-J(t-s)} D(s-u) ds \right| du \\ \leq\ & e^{-Jt} + \int_0^t \left[Je^{-J(t-u)} - \int_u^t e^{-J(t-s)} D(s-u) ds \right] du \\ =\ & e^{-Jt} + \int_0^t Je^{-J(t-u)} - \int_0^t T(u;t)\, du \\ =\ & 1 - \int_0^t T(u;t)\, du \leq 1; \end{aligned}$$

i.e., (25) holds true. □

The lemma is only the first brief step in the study of conditions to ensure (25). A lengthy treatment of conditions for the linear, nonlinear, and forced equation will be presented later.

Note This is a corrected file. In the original published version we always let M be a closed bounded convex subset of BC. In a correction published in Nonlinear Anal. **100**(2014) 97-98 we note that M must be a ball. All of those results were based on a theorem in [8] which has a correction in press and should appear in the same journal in the first issue in 2014. With that change it was necessary to delete the old conclusion of Theorem 5.1 which said that the fixed point converged to zero.

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