A SCHAUDER-TYPE FIXED POINT THEOREM

T. A. BURTON AND BO ZHANG

Abstract. In this brief note we study Schauder's second fixed point theorem in the space $(BC, \|\cdot\|)$ of bounded continuous functions $\phi : [0, \infty) \to \Re^n$ with a view to reducing the requirement that there is a compact map to the requirement that the map is locally equicontinuous. Several examples are given, both motivating and applying the theory.

Keywords: Schauder's theorem, fixed points, fractional differential equations, nonlinear integral equations
Mathematics Subject Classification: 47H10, 45G10, 34A08

1. Introduction

This note represents an attempt to draw Schauder's fixed point theorem in a Banach space $(BC, \|\cdot\|)$ of bounded continuous functions $\phi : [0, \infty) \to \Re^n$ with the supremum norm closer to the much simpler Brouwer's fixed point theorem for R^n.

In the simplest form Brouwer's fixed point theorem states that any continuous mapping of the closed ball in Euclidean n-space into itself has a fixed point. The theory of retracts extends this ball to any closed bounded (therefore compact) convex nonempty set.

Schauder took two steps toward extending Brouwer's result to a Banach space. First, he removed the boundedness condition. Then he added a condition which has proved to be very difficult to verify in one of the most important cases: That is when M consists of continuous functions $\phi : [a, \infty) \to \Re^n$ and $P : M \to M$ with P continuous and M closed, convex, and nonempty. He required that either M be compact or P map M into a compact subset $K \subset M$ of the Banach space. These conditions are not too stringent when M consists of continuous functions $\phi : [a, b] \to \Re^n$ since we can frequently arrange matters so that PM is equicontinuous and then apply the Ascoli-Arzela theorem. But the compactness can be very difficult to establish when we are working on the entire interval $[0, \infty)$.

Corduneanu [4, p. 62] took an interesting step in the same direction as did Schauder. First, he restored the boundedness assumption. Next, he asked that when the functions in M are restricted to any finite subdomain then PM is equicontinuous. His final condition is that for large t then M is essentially frozen in time: For each $\epsilon > 0$ there is a T such that $|f(t) - f(\infty)| < \epsilon$ for all $t \geq T$ and all f in M. In this note we will accept most of the requirements of Corduneanu, but we will eliminate any idea of demanding uniform limits at infinity.

We have also recently studied this type of problem for specific integral equations [1] which generated maps that were equicontinuous on $[0, \infty)$. A brief correction is being processed by the journal asking that the mapping set be a ball or a certain retract.

The exact statements of the classical theorems discussed here can be found in Smart [6].

2. The fixed point theorem

We begin with the Banach space $(BC, \|\cdot\|)$ as described in Section 1. It is assumed that there is a convex, nonempty, bounded subset M of BC and a continuous mapping $P : M \to M$ with the property that for each $T > 0$ the functions in PM restricted to the domain $[0, T]$ are equicontinuous. We will say that such a set is **locally equicontinuous**. To avoid possible confusion, we remark that there are different conventions concerning the definition of equicontinuity. If, for example, we consult Royden and Fitzpatrick [5, pp. 207-8] we see that some authors would call this equicontinuity, not local equicontinuity.

We will also be introducing a new space as follows. Let $g : [0, \infty) \to [1, \infty)$ be an arbitrary continuous strictly increasing function with $g(t) \to \infty$ as $t \to \infty$ and define a Banach space $(W, |\cdot|_g)$ of continuous functions $\psi : [0, \infty) \to \Re^n$ with the property that

$$|\psi|_g = \sup_{0 \le t < \infty} \frac{|\psi(t)|}{g(t)} < \infty$$

which will be used as follows. In the example below we have constructed a set

$$M = \{\phi \in BC |\; a \le \phi(t) \le b,\, |\phi(t) - \phi(s)| \le |t - s|, t,\, s \ge 0\}$$

which would contain a fixed point for any continuous self-map, say P. But in many problems we wish to have a fixed point with a particular property so we add that to the mapping set. Have we added so much that there may no longer be a fixed point for certain continuous self-maps? The test we offer is to see if the set is closed in the weighted norm. If for this mapping PM resides in a subset of M which is closed in the weighted norm then we will still have a fixed point. By "the g-norm closure of a set $E \subset BC$" we mean the closure of E in $(W, |\cdot|_g)$. The next example illustrates the test.

Example 1. Let $(BC, \|\cdot\|)$ be the Banach space of bounded continuous functions $\phi : [0, \infty) \to R$ with the supremum norm. Let $M \subset BC$ be defined as

$$M = \{\, x \in BC \big|\; \|x\| \le 1,\ |x(t) - x(s)| \le |t - s|,\ \forall\, t, s \ge 0,\ \lim_{t\to\infty} x(t) = 1 \,\}$$

Then M is a convex, nonempty, bounded subset of BC. Now define $P : M \to M$ by

$$(Px)(t) = \frac{t}{t+1} x(t)$$

for any $x \in M$. Have we asked too much of M?

We see that P is continuous on M and $PM \subset M$. We also notice that PM is locally equicontinuous. Thus, all conditions of the up coming Theorem 2.1a are satisfied except the condition that the "g-norm closure of PM is in M." We see that P has no fixed point in M. Note that, in this case, the g-norm closure of PM is not in M. To see this, let K be the closure of PM in $(W, |\cdot|_g)$ and define $\phi_n \in BC$ by

$$\phi_n(t) = t - n \quad \text{for } \; n \le t \le n + 1,$$

with $\phi_n(t) = 0$ for $t < n$ and $\phi_n(t) = 1$ for $t > n + 1$. Then $\phi_n \in M$. We see that $|P\phi_n - \phi|_g \to 0$ as $n \to \infty$ where $\phi \equiv 0$ on $[0, \infty)$. This implies that $\phi \in K$. However, $\phi \notin M$. Thus, $K \not\subset M$. That last condition which we added to M was simply too much to guarantee a fixed point for every continuous mapping of M into M.

We shall use the notation $\|\phi\|^{[0,\,T\,]} = \sup_{0 \le t \le T} |\phi(t)|$ for any continuous function ϕ defined on $[0, T]$. There are two equivalent ways to state the result and we offer both of them. The difference is that the second form states one way of showing that the $g-$norm closure of PM is in M.

Theorem 2.1a. Let M be a convex, nonempty, bounded subset of BC and let $P : M \to M$ be continuous, and PM locally equicontinuous. If, in addition, the g-norm closure of PM is in M, then there exists a point $\phi \in M$ with $P\phi = \phi$.

Theorem 2.1b. Let M be a convex, nonempty, bounded subset of BC and let $P : M \to M$ be continuous, and PM locally equicontinuous. If, in addition, for any sequence $\{x_n\}$ in M, $\|Px_n - x\|^{[0,\,T\,]} \to 0$ as $n \to \infty$ for each fixed $T > 0$ implies that $x \in M$, then there exists a point $\phi \in M$ with $P\phi = \phi$.

Proof. By Lemma 2.1 and Lemma 2.2 given below, we know that P is also continuous on M in $(W, |\cdot|_g)$ and PM resides in a compact subset K of $(W, |\cdot|_g)$. Now let K be the closure of PM in $(W, |\cdot|_g)$. We show that $K \subset M$. In fact, for each $x \in K$, there exists a sequence $\{x_n\}$ in M with $|Px_n - x|_g \to 0$ as $n \to \infty$. For each $T > 0$, we have

$$\|Px_n - x\|^{[0, T]} \leq g(T)|Px_n - x|_g \to 0 \quad \text{as} \quad n \to \infty.$$

This then implies that $x \in M$ by the assumption. Thus, $K \subset M$. Applying Schauder's second fixed point theorem to $P : M \to K \subset M$ in $(W, |\cdot|_g)$, we obtain that there exists a point $\phi \in M$ with $P\phi = \phi$. The proof is complete. □

Remark 1. The conditions in the theorems above are very precise in the sense that P may not have a fixed point if the g-norm closure of PM is not in M (see Example 1). Also, the equicontinuity condition on PM is necessary for PM being pre-compact in $(W, |\cdot|_g)$. This fact is well-known for theorems of Ascoli-Arzela type (see Royden and Fitzpatrick [5, p. 209]).

Lemma 2.1. *If P is continuous on M in the supremum norm, $\|\cdot\|$, then it is also continuous on M in the weighted norm, $(W, |\cdot|_g)$.*

Proof. To say that P is continuous at $\phi \in M$ in the weighted norm is to say that for each $\epsilon > 0$ there is a $\delta > 0$ such that $\eta \in M$ and $|\phi - \eta|_g < \delta$ implies that $|P\phi - P\eta|_g < \epsilon$. Let $\epsilon > 0$ be given.

First notice that there exists $K^* > 0$ such that if $\phi, \eta \in M$ then $\|P\phi\| \leq K^*$ and $\|P\eta\| \leq K^*$ so we select $T > 0$ so that $g(T) > 2K^*/\epsilon$. It then follows that

$$\sup_{t \geq T} \frac{|(P\phi)(t) - (P\eta)(t)|}{g(t)} \leq \frac{2K^*}{g(T)} < \epsilon.$$

Thus, we restrict our work to $0 \leq t \leq T$.

Now, P is continuous on M in the supremum norm by assumption so for the given $\phi \in M$ and $\epsilon > 0$ there is a $\delta > 0$ such that $\eta \in M$ and $\|\phi - \eta\| < \delta g(T)$ implies that $\|P\phi - P\eta\| < \epsilon$. Thus, $|\phi - \eta|_g < \delta$ implies that $|\phi(t) - \eta(t)| < \delta g(t) < \delta g(T)$ for $0 \leq t \leq T$ so

$$\sup_{0 \leq t \leq T} \frac{|(P\phi)(t) - (P\eta)(t)|}{g(t)} < \epsilon$$

□

Lemma 2.2. *The set PM resides in a compact subset of $(W, |\cdot|_g)$.*

Proof. Since the g-norm closure of PM is in M, the proof is essentially identical to that of Theorem 3.3 in [2]. We use Ascoli's theorem and a diagonalization process in the classical way to show that any sequence in PM has a subsequence converging uniformly on compact sets to a continuous function on $[0, \infty)$ and that function is in M. The local equicontinuity is used as follows. The diagonalization process takes place on intervals $[0, n]$; the equicontinuity holds on each of these and so the Ascoli theorem establishes the convergence on that interval. □

Corollary 1. Let M be a convex, nonempty, bounded subset of BC and let $P : M \to M$ be continuous, and PM locally equicontinuous. If M is closed in $(W, |\cdot|_g)$, then there exists a point $\phi \in M$ with $P\phi = \phi$.

Proof. Since $PM \subset M$ and M is closed in $(W, |\cdot|_g)$, we see that if K is the closure of PM in $(W, |\cdot|_g)$, then $K \subset M$. Thus, all conditions of Theorem 2.1a are satisfied. □

A special case of Example 2 shows that M is closed in $(W, |\cdot|_g)$ if it is a closed ball in BC.

Corollary 2. Let M be a closed ball in BC and let $P : M \to M$ be continuous, and PM locally equicontinuous. Then there exists a point $\phi \in M$ with $P\phi = \phi$.

In the counterexample given above the functions all had the same limit at infinity, but they converged to that limit at varying rates. There are problems in functional differential equations concerning uniform asymptotic stability or uniform ultimate boundedness in which we need to show that the functions in the mapping set enter a prescribed small set for large times. The next example shows that our theorem will work in these cases if the functions enter that small set in a uniform manner.

Example 2. The set

$$M = \{x \in BC | \, \|x\| \le L \text{ and } |x(t)| \le B \text{ for } t \ge T\}$$

is closed in $(W, |\cdot|_g)$.

Proof. Let $x_n \in M$. Suppose that there exists $x \in (W, |\cdot|_g)$ with $|Px_n - x|_g \to 0$ as $n \to \infty$. Then for each $\ell \ge 0$, we have

$$\|Px_n - x\|^{[0,\ell]} \le g(\ell)|Px_n - x|_g \to 0 \text{ as } n \to \infty.$$

This implies that $\|x\| \le \sup_{n\ge1} \|Px_n\| \le L$. Now for each $\ell \ge T$, we also have

$$\|Px_n - x\|^{[T,\ell]} \le g(\ell)|Px_n - x|_g \to 0 \text{ as } n \to \infty$$

so that $\|x\|^{[T,\infty)} \le \sup_{n\ge1} \|Px_n\|^{[T,\infty)} \le B$. Thus, $x \in M$. This proves that M is closed in $(W, |\cdot|_g)$. □

Example 3. The set

$$M = \{x \in BC | \, \|x\| \le L, \ |x(t_1) - x(t_2)| \le K\, |t_1 - t_2|^q, \ \forall\, t_1, \ t_2 \ge 0\, \}$$

is closed in $(W, |\cdot|_g)$, where $0 < q \le 1$.

Proof. Let $x_n \in M$. If there exists $x \in (W, |\cdot|_g)$ with $|Px_n - x|_g \to 0$ as $n \to \infty$, then $\|x\| \le L$. For any $t_1, \ t_2 \ge 0$, we have

$$\begin{aligned} |x(t_1) - x(t_2)| &\le |(Px_n)(t_1) - x(t_1)| + |(Px_n)(t_1) - (Px_n)(t_2)| + |(Px_n)(t_2) - x(t_2)| \\ &\le g(t_1)|Px_n - x|_g + K\, |t_1 - t_2|^q + g(t_2)|Px_n - x|_g. \end{aligned}$$

Letting $n \to \infty$, we obtain $|x(t_1) - x(t_2)| \le K |t_1 - t_2|^q$. This implies that $x \in M$. Therefore, M is closed in $(W, |\cdot|_g)$. The proof is complete. □

Example 4. Consider the fractional differential equation of Caputo type

$$ {}^cD^q x = -a(t)x^3(t) + b(t)x^3(t - r(t)), \ \ x(0) = x_0, \ \ 0 < q < 1, \tag{1}$$

with $a, b, r : [0, \infty) \to \Re$ continuous. See [3] for background and definitions. Suppose that

(i) $a(t)$ is bounded on $[0, \infty)$.

(ii) $a(t) - |b(t)| \ge \delta$ for all $t \ge 0$ and a constant $\delta > 0$.

(iii) $r(t) \ge 0$ for all $t \ge 0$.

Then the zero solution of (1) is stable.

Proof. Choose a constant $\eta > 0$ with $\sup_{t\ge0} a(t) < \eta$ for all $t \ge 0$ and define

$$C(t) = \frac{\eta}{\Gamma(q)} t^{q-1}$$

Then the resolvent R satisfies

$$R(t) = C(t) - \int_0^t C(t - s)R(s)ds$$

This resolvent R is completely monotone on $(0,\infty)$. Moreover,

$$0 \le R(t) \le C(t),\ tR(t) \to 0 \text{ as } t \to \infty, \text{ and } \int_0^\infty R(s)ds = 1.$$

If we write (1) as

$$\begin{aligned} {}^cD^q x &= -a(t)x^3(t) + b(t)x^3(t-r(t)) \\ &= -\eta x(t) + \eta[x - x^3] + [\eta - a(t)]x^3 + b(t)x^3(t-r(t)) \end{aligned}$$

then the solution $x(t)$ of (1) satisfies

$$\begin{aligned} x(t) &= z(t) + \int_0^t R(t-s)[x(s) - x^3(s)]ds \\ &+ \int_0^t R(t-s)\left(1 - \frac{a(t)}{\eta}\right)x^3(s)ds + \int_0^t R(t-s)\frac{b(t)}{\eta}x^3(s-r(s))ds =: (Px)(t) \end{aligned}$$

where $z(t) = x_0(1 - \int_0^t R(s)ds)$.

Let $0 < \varepsilon < \sqrt{3}/3$. We may assume that $\eta \ge 1$ and $0 < \delta < 1$ so that $\delta\varepsilon^3/\eta < \varepsilon$. Now set $m_0 = \inf\{s - r(s) : s \ge 0\}$. Let $\psi : [m_0, 0] \to \Re$ be a given continuous initial function with $\|\psi\| < \delta\varepsilon^3/\eta$. Define

$$M = \{\phi \in BC|\ \|\phi\| \le \varepsilon\}.$$

For the natural mapping defined above with $x(s) = \psi(s)$ for $s \le 0$, we can show that $P : M \to M$. To see this, we observe that $r - r^3$ is increasing on $[0,\varepsilon]$ and apply (ii) to obtain

$$\begin{aligned} |(Px)(t)| &\le |z(t)| + (\varepsilon - \varepsilon^3) + \varepsilon^3 \int_0^t R(t-s)\left(1 - \frac{a(t)}{\eta} + \frac{|b(t)|}{\eta}\right)ds \\ &\le |\psi(0)| + (\varepsilon - \varepsilon^3) + \varepsilon^3(1 - \delta/\eta) < \varepsilon \end{aligned}$$

if $\|\psi\| < \delta\varepsilon^3/\eta$. It is clear that PM is equicontinuous on $[0,\infty)$ and the g-norm closure of PM is in M by Corollary 2. By Theorem 2.1a, P has a fixed point $x \in M$ which is a solution of (1). Thus, the zero solution of (1) is stable. □

Example 5. Consider the scalar integral equation

$$x(t) = f(t) + \int_0^t R(t-s)a(s)g(x(s))ds.$$

The kernel is used to obtain locally equicontinuous maps even when $a(t)$ is unbounded, but integrable in a certain sense. Here are our assumptions.

(i) R is positive and decreasing with $R(t) \le t^{q-1}$ for some $q \in (0,1)$.

(ii) $a : [0,\infty) \to \Re$ is continuous with $a \in L^p[0,\infty)$ for some $p > 1/q$.

(iii) $g : \Re \to \Re$ is continuous, $|g(x)| \to \infty$ as $|x| \to \infty$, and $g(x)/x \to 0$ as $|x| \to \infty$.

(iv) $f : [0,\infty) \to \Re$ is continuous with $\sup_{t\ge 0}|f(t)| < \infty$.

Let p be given in (ii) and define $p^* = p/(p-1)$. Then $(1/p^*) + (1/p) = 1$. Observe that $p > 1/q$ implies $p^*(1-q) < 1$ and therefore, $R \in L^{p^*}[0,\infty)$. This results in

$$\int_0^t R(t-s)|a(s)|ds \le |R|_{p^*}|a|_p =: K.$$

For $x \ge 0$, we define

$$g^*(x) = \max\{|g(x)|, |g(-x)|\}.$$

Next we choose a number $\alpha > 0$ with $0 < K\alpha < 1$. Since $|g(x)| \to \infty$ and $g(x)/x \to 0$ as $|x| \to \infty$, we can assert the existence of a number $b > 0$ with

$$g^*(b)/b \leq \alpha < 1, \ \ \|f\| + K\alpha b < b, \ \text{ and } \ |g(u)| \leq g^*(b) \ \text{ for } \ |u| \leq b.$$

There is a continuous solution in

$$M = \{\phi \in BC| \ \|\phi\| \leq b\}.$$

Proof. Note that $P : M \to M$ where P is the natural mapping defined by the integral equation. To see this, let $\phi \in M$ and use Hölder's inequality to obtain

$$\begin{aligned} |(P\phi)(t)| &\leq \|f\| + \int_0^t |R(t-s)a(s)|g^*(b)ds \\ &\leq \|f\| + Kg^*(b) \leq \|f\| + K\alpha b \leq b. \end{aligned}$$

For such mappings it is shown in [3] that if a is bounded and continuous, then the integral part of PM is equicontinuous on the entire interval $[0, \infty)$. But a is unbounded so we have local equicontinuity of PM on $[0, \infty)$.

We now show that P is continuous, so that our theorem will give a solution in M. To that end, if $\phi, \eta \in M$ then

$$|(P\phi)(t) - (P\eta)(t)| \leq \int_0^t R(t-s)|a(s)||g(\phi(s)) - g(\eta(s))|ds.$$

But g is uniformly continuous on $[-b, b]$ so for a given $\epsilon > 0$ there is a $\delta > 0$ so that $\|\phi - \eta\| < \delta$ implies that $|g(\phi(t)) - g(\eta(t))| \leq \epsilon/K$. That will establish the continuity of P on M. Since M is a closed ball in BC and $PM \subset M$, it follows from Corollary 2 that the g-norm closure of PM is in M. This proves that the integral equation has a continuous solution on $[0, \infty)$ and it lies in M. □

References

[1] T. A. Burton, Fractional equations and a theorem of Brouwer-Schauder type, *Fixed Point Theory* **14** (2013), No. 1, 91-96.

[2] T. A. Burton and Bo Zhang, Fractional equations and generalizations of Schaefer's and Krasnoselskii's fixed point theorems, *Nonlinear Anal.: TMA* **75**, (2012), 6485-6495.

[3] T. A. Burton and Bo Zhang, Fixed points and fractional differential equations: Examples, *Fixed Point Theory*, **14**(2013), No. 2, 313-326.

[4] C. Corduneanu, *Integral Equations and Stability of Feedback Systems*, Academic Press, New York, 1973

[5] H. L. Royden and P. M. Fitzpatrick, *Real Analysis*, Prentice Hall, 2010.

[6] D. R. Smart, *Fixed Point Theorems*, Cambridge University Press, 1980.

Northwest Research Institute, 732 Caroline St., Port Angeles, WA 98362
E-mail address: taburton@olypen.com

Department of Mathematics and Computer Science, Fayetteville State University, Fayetteville, NC 28301
E-mail address: bzhang@uncfsu.edu