ASYMPTOTICALLY PERIODIC SOLUTIONS OF FRACTIONAL DIFFERENTIAL EQUATIONS (APSOFDE-5.TEX)

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ABSTRACT. In three recent papers investigators have shown that a linear fractional differential equation can not have a periodic solution. This raises two fundamental questions: What are the properties of the out-put function if the in-put function is periodic? What are the properties of perturbations that will leave the out-put function unchanged? We answer both questions here. The out-put function is asymptotically periodic and it is unchanged by perturbations which are $L^1[0,\infty)$ and by perturbations which tend to zero as $t \to \infty$ with these perturbations applied simultaneously in the damping and the forcing terms. We also find a limiting equation which this periodic function satisfies. The methods used include limiting equation techniques and fixed point methods involving both contractions and Krasnoselskii-Schaefer type.

1. Introduction

We consider a fractional differential equation of Caputo type

$$^{c}D^{q}x = -u(t,x(t)), \quad 0 < q < 1,$$

where $u(t,x)$ is continuous and there is a $T > 0$ with $u(t+T,x) = u(t,x)$ for all $(t,x) \in \Re \times \Re$. When $u(t,x)$ is continuous then this equation is immediately inverted as the very familiar integral equation ([15, p. 54], [10, pp. 78, 86, 103])

$$x(t) = x(0) - \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}u(s,x(s))ds$$

where Γ is the gamma function. This equation has a unique solution as seen in [1] or [20]

In three recent papers investigators ([19],[18],[11]) have offered proofs that certain equations of this type can not have periodic solutions. This is important, but hardly surprising. Even an ordinary differential equation of the form

$$z'(t) = Az(t) + \int_0^t C(t-s)z(s)ds + f(t)$$

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with A constant and f periodic will **seldom** have a periodic solution. The reason for this is that when $x(t)$ is a periodic function, then only under very special circumstances will it be true that $X(t) := \int_0^t C(t-s)x(s)ds$ is periodic. This is easily checked by computing $X(t+nT)$ and letting $n \to \infty$. The problem seems to have been first studied in Burton [3] and continued in [5, pp. 94-96].The periodicity depends on a special orthogonal property discussed by Lakshmikantham and Rao [14, pp. 120-123].

With this introduction, we explain the term "seldom" by noting that $z = \cos t + \sin t$ solves

$$z' = az + b\int_0^t e^{-(t-s)}z(s)ds - (1+a+b)\sin t + (1-a)\cos t.$$

Notice, in particular, the $L^1[0,\infty)$ kernel. The natural periodicity occurs in $\int_{-\infty}^t C(t-s)x(s)ds$ when x is periodic, but then only if C is an $L^1[0,\infty)$ function. That never happens with the kernel $C(t) = \frac{1}{\Gamma(q)}t^{q-1}$.

This problem was discussed for integral equations in Burton [6, p. 58] and the natural solution in such problems is an asymptotically periodic solution. But, again, that depends on an L^1 kernel, a property we will obtain here by a transformation to a new kernel, $R(t-s)$, which is positive, completely monotone, and $\int_0^\infty R(t)dt = 1$.

2. Basic theory

In this paper we always contrive (see Section 6) to begin with

$$(1)\qquad {}^cD^q x(t) = -[x(t) + G(t,x(t))] + f(t), \quad 0 < q < 1, \quad x(0) \in \Re$$

with $G : [0,\infty) \times \Re \to \Re$ and $f : [0,\infty) \to \Re$ both being continuous. This equation is then inverted as

$$(2)\qquad x(t) = x(0) - \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}[x(s) + G(s,x(s)) - f(s)]ds.$$

Denote the kernel by

$$(3)\qquad C(t) = \frac{1}{\Gamma(q)}t^{q-1}$$

so that for any $T > 0$ we have the critical property that

$$\int_0^T |C(u)|du < \infty.$$

Following Miller [17, pp. 193-22] we note that $C(t)$ is completely monotone on $(0,\infty)$ in the sense that $(-1)^k C^{(k)}(t) \geq 0$ for $k = 0, 1, 2, ...$ and $t \in (0,\infty)$. Moreover $C(t)$ satisfies the conditions of Miller's Theorem 6.2 on p. 212. That theorem states that if the resolvent equation for the kernel C is

$$(4)\qquad R(t) = C(t) - \int_0^t C(t-s)R(s)ds$$

then that resolvent kernel, R, satisfies

$$0 \le R(t) \le C(t) \text{ for all } t > 0 \text{ so as } t \to \infty \text{ then } R(t) \to 0 \tag{5}$$

and that

$$C \notin L^1[0,\infty) \quad \Longrightarrow \int_0^\infty R(s)ds = 1. \tag{6}$$

Continuing on to [17, pp. 221-224 (Theorem 7.2)] we see that R is also completely monotone.

Next, under the conditions here, it is shown in Miller [17, pp. 191-207] that (2) can be decomposed into

$$z(t) = x(0) - \int_0^t C(t-s)z(s)ds \tag{7}$$

with

$$z(t) = x(0) - \int_0^t R(t-s)x(0)ds = x(0)[1 - \int_0^t R(s)ds]$$

and, having found $z(t)$, then the solution $x(t)$ of (2) solves

$$x(t) = z(t) - \int_0^t R(t-s)G(s,x(s))ds + \int_0^t R(t-s)f(s)ds. \tag{8}$$

Notice that $z(t) \to 0$ as $t \to \infty$.

The kernel in (2) is not integrable on $[0,\infty)$, but in (8) it is replaced, not only by an integrable kernel, but the value of the integral is one and the new kernel is also completely monotone.

3. The main motivation

In order to see what can be expected from periodic forcing we begin with the equation

$$^cD^q x = -x(t) + f(t), \quad 0 < q < 1, \quad f(t+T) = f(t) \tag{9}$$

for some fixed $T > 0$ and all t with f continuous. This can be inverted as

$$x(t) = x(0) - \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}[x(s) - f(s)]ds$$

which we write as

$$x(t) = x(0) - \int_0^t C(t-s)[x(s) - f(s)]ds.$$

with

$$C(t) = \frac{1}{\Gamma(q)}t^{q-1} \quad \text{for } \ t > 0. \tag{10}$$

We decompose this as

$$z(t) = x(0) - \int_0^t C(t-s)z(s)ds$$

so that

$$z(t) = x(0) - \int_0^t R(t-s)x(0)ds$$

$$= x(0)[1 - \int_0^t R(s)ds] = x(0)\int_t^\infty R(s)ds$$

and

$$x(t) = z(t) + \int_0^t R(t-s)f(s)ds. \tag{11}$$

where, again, the resolvent satisfies

$$R(t) = C(t) - \int_0^t C(t-s)R(s)ds$$

having properties (5) and (6).

Theorem 3.1. *The function*

$$y(t) =: \int_{-\infty}^t R(t-s)f(s)ds \tag{12}$$

is T-periodic and for any $x(0) \in \Re$ the solution $x(t)$ of (11) satisfies

$$x(t+nT) \to y(t) \quad \text{as } n \ \to \infty \tag{13}$$

uniformly on any compact subset of $\Re$.

Proof. Let $K > 0$ be fixed and n a positive integer with $nT > K$. Then for $t \in [-K, K]$, we have

$$|z(t+nT)| = |x_0| \int_{t+nT}^\infty R(u)du \le |x_0| \int_{nT-K}^\infty R(u)du$$

and

$$\left| \int_{-\infty}^{-nT} R(t-s)f(s)ds \right| \le \int_{-\infty}^{-nT} R(t-s)ds\|f\|$$

$$= \|f\| \int_{t+nT}^\infty R(u)du \le \|f\| \int_{nT-K}^\infty R(u)du.$$

Replacing t by $t + nT$ in (11), we see that

$$x(t+nT) = z(t+nT) + \int_{-nT}^t R(t-s)f(s)ds.$$

Next, we write

$$y(t) = \int_{-nT}^t R(t-s)f(s)ds + \int_{-\infty}^{-nT} R(t-s)f(s)ds$$

and apply the two inequalities above to obtain

$$\begin{aligned}
&|x(t+nT)-y(t)| \\
&= \left| z(t+nT) - \int_{-\infty}^{-nT} R(t-s)f(s)ds \right| \\
&\le (|x_0| + \|f\|) \int_{nT-K}^{\infty} R(u)du \to 0
\end{aligned}$$

as $n \to \infty$. This implies that $x(t+nT) \to y(t)$ uniformly on $[-K, K]$. Moreover, a change of variable in (12) shows that $y(t+T) = y(t)$. Thus, we have the desired conclusion. □

Can we do better? Can we prove that $x(t)$ is a periodic solution for an appropriate $x(0)$? First, we might invert the fractional equation using the Riemann inversion as

$$x(t) = -\int_{-\infty}^{t} C(t-s)[x(s) - f(s)]ds.$$

This would symbolically yield a mapping

$$(Z\phi)(t) = -\int_{-\infty}^{t} C(t-s)[\phi(s) - f(s)]ds$$

so that ϕ periodic implies $Z\phi$ periodic. That conclusion is wrong because the integral may not converge.

Theorem 3.2. *Solutions of (9) are uniformly bounded on $[0, \infty)$ and all of them converge to the periodic function $y(t)$ defined in (12). Thus, we say that all solutions are asymptotically periodic and that $y(t)$ is a global attractor and an asymptotic T-periodic solution of (9).*

Proof. Let $x(t)$ be a solution of (9) with $x(0) = x_0$. Then we have (11) and so,

$$\begin{aligned}
|x(t)| &\le |z(t)| + \int_0^t R(t-s)|f(s)|ds \\
&\le |x_0|[1 - \int_0^t R(u)du] + \|f\| \int_0^t R(t-s)ds \\
&\le B_1 + \|f\| =: B_2
\end{aligned} \tag{14}$$

if $|x_0| \le B_1$. Thus, solutions of (9) are uniformly bounded on $[0, \infty)$. Next, we write $x(t)$ as

$$\begin{aligned}
x(t) &= y(t) + z(t) - \int_{-\infty}^{0} R(t-s)f(s)ds \\
&=: y(t) + \phi(t).
\end{aligned}$$

Since
$$|\phi(t)| \le |z(t)| + \|f\| \int_t^\infty R(u)du \to 0 \quad \text{as} \quad t \to \infty,$$
and $y(t) \in P_T$, we see that $y(t)$ is an asymptotic T-periodic solution of (9) to which all other solutions converge. This completes the proof. □

Theorem 3.3. *If (9) has a T-periodic solution $\widetilde{x}(t)$, then $\widetilde{x}(t) = y(t)$ and*
$$\psi(t) = \int_t^\infty R(u)f(t-u)du$$
is a solution of the homogeneous equation
$$^cD^q x = -x(t). \tag{15}$$

Proof. If (9) has a T-periodic solution $\widetilde{x}(t)$, then it must be that $\widetilde{x}(t) = y(t)$ for all $t \ge 0$. But $\int_0^t R(t-s)f(s)ds$ is also a solution of (9) with initial value $x_0 = 0$ and so, the difference
$$\int_{-\infty}^0 R(t-s)f(s)ds$$
is a solution of (15). Let $s = t - u$ to complete the proof. □

Remark 3.4. *Under the condition of Theorem 3.3, we have $\psi(t)$ a solution of (15) while every solution of (15) can be expressed as $b[1 - \int_0^t R(u)du] = b\int_t^\infty R(u)du$ for some constant b, and hence*
$$b\int_t^\infty R(u)du = \int_t^\infty R(u)f(t-u)du \quad \text{for} \quad t \ge 0.$$
It is clear that if $f(t) \equiv f_0$, a constant, then $\widetilde{x}(t) = f_0$ is a constant solution of (9) and $f_0 \int_t^\infty R(u)du$ is a solution of (15).

4. A limiting equation for (9)

For decades investigators have studied the counterpart of our problem here for integro-differential equations and a synopsis can be found in [4, pp. 90-97]. Let A be an $n \times n$ constant matrix, $B \in L^1[0,\infty)$ be an $n \times n$ matrix of continuous functions, and p be a column vector function of continuous $T-$periodic functions. If Z is the principal matrix solution of
$$x'(t) = Ax(t) + \int_0^t B(t-s)x(s)ds + p(t)$$
with $Z \in L^1[0,\infty)$, then
$$\phi(t) = \int_{-\infty}^t Z(t-s)p(s)ds$$

is a $T-$periodic solution of

$$y'(t) = Ay(t) + \int_{-\infty}^{t} B(t-s)y(s)ds + p(t),$$

called a limiting equation. We now find a limiting equation for our $y(t)$; it is an unusual fractional differential equation and it relates to our early comment about writing the Riemann inversion of (1), just before Theorem 3.2.

We now show that $y(t)$ in (12) is a T-periodic solution of

$$^{c}D_{+}^{q}x(t) = -x(t) + f(t) \tag{16}$$

if $f \in P_T$ with f' continuous, where $^{c}D_{+}^{q}x$ is the Caputo fractional derivative of x on $\Re$ (see [12, pp. 87]) with

$$^{c}D_{+}^{q}x(t) = \frac{1}{\Gamma(1-q)} \int_{-\infty}^{t} (t-s)^{-q}x'(s)ds. \tag{17}$$

We point out that by Dirichlet's test, the integral on the right-hand side of (17) converges if $x \in P_T$ with x' continuous. We also view (16) as the limiting equation of (9) (see the proof below).

Theorem 4.1. *If $f \in P_T$ and f' is continuous on $\Re$, then $y(t) = \int_{-\infty}^{t} R(t-s)f(s)ds$ is a T-periodic solution of (16).*

Proof. Let $x(t)$ be a solution of (9) with $x(0) = x_0$. Then we have

$$x(t) = z(t) + \int_0^t R(t-s)f(s)ds.$$

We first claim that $x'(t)$ exists and is continuous for $t > 0$. In fact, it follows from $z(t) = [1 - \int_0^t R(u)du]x_0$ that $z'(t) = -R(t)x_0$. Also, we have

$$\begin{aligned}\frac{d}{dt}\int_0^t R(t-s)f(s)ds &= \frac{d}{dt}\int_0^t R(u)f(t-u)du \\ &= R(t)f(0) + \int_0^t R(u)f'(t-u)du\end{aligned}$$

for $t > 0$ and thus,

$$x'(t) = -R(t)x_0 + R(t)f(0) + \int_0^t R(u)f'(t-u)du \tag{18}$$

exists and is continuous for $t > 0$. Writing $y(t) = \int_0^{\infty} R(u)f(t-u)du$, we see that

$$y'(t) = \int_0^{\infty} R(u)f'(t-u)du = \int_{-\infty}^{t} R(t-s)f'(s)ds$$

for all $t \in \Re$.

Next, let $K > 0$ be fixed. Then for $t \in [-K, K]$ and $nT > K$, we have

$$x'(t+nT) = -R(t+nT)x_0 + R(t+nT)f(0) + \int_{-nT}^{t} R(t-s)f'(s)ds.$$

Subtract $y'(t)$ from both sides of the equation above to obtain

$$\begin{aligned} x'(t+nT) - y'(t) &= -R(t+nT)x_0 + R(t+nT)f(0) \\ &\quad - \int_{-\infty}^{-nT} R(t-s)f'(s)ds. \end{aligned}$$

We then have

$$\begin{aligned} &|x'(t+nT) - y'(t)| \\ &\leq R(nT-K)|(|x_0| + |f(0)|) + \|f'\| \int_{nT-K}^{\infty} R(u)du. \end{aligned}$$

This implies that $x'(t+nT) \to y'(t)$ uniformly in t on $[-K, K]$. We know from Theorem 3.1 that $x(t+nT) \to y(t)$ uniformly in t on $[-K, K]$. Replacing t by $t+nT$ in (9) with $nT > -t$, we obtain

$$^cD^q x(t+nT) = -x(t+nT) + f(t). \tag{19}$$

It is clear that the right-hand side converges to $-y(t) + f(t)$ as $n \to \infty$ for each $t \in \Re$.

We now show that $^cD^q x(t+nT) \to {}^cD^q_+ y(t)$ as $n \to \infty$. Let $t > -K$ be fixed and $nT > K$. From the definition of $^cD^q x$, we have

$$\begin{aligned} ^cD^q x(t+nT) &= \frac{1}{\Gamma(1-q)} \int_0^{t+nT} (t+nT-s)^{-q} x'(s)ds \\ &\qquad \text{(change variable by } u = s - nT\text{)} \\ &= \frac{1}{\Gamma(1-q)} \int_{-nT}^{t} (t-u)^{-q} x'(u+nT)du \\ &= \frac{1}{\Gamma(1-q)} \int_{-K}^{t} (t-u)^{-q} x'(u+nT)du \\ &\quad + \frac{1}{\Gamma(1-q)} \int_{-nT}^{-K} (t-u)^{-q} x'(u+nT)du \\ &=: I_1 + I_2. \end{aligned}$$

Since $x'(u+nT) \to y'(u)$ as $n \to \infty$ uniformly on $[-K, t]$, we see that

$$I_1 \to \frac{1}{\Gamma(1-q)} \int_{-K}^{t} (t-u)^{-q} y'(u)du \ \text{ as } \ n \to \infty. \tag{20}$$

We also observe that

$$\begin{aligned}
I_2 &= \frac{1}{\Gamma(1-q)} \int_{-nT}^{-K} (t-u)^{-q} x'(u+nT)du \\
&= \frac{1}{\Gamma(1-q)} \Bigg[(t-u)^{-q} x(u+nT) \Big|_{-nT}^{-K} \\
&\qquad\qquad - q \int_{-nT}^{-K} (t-u)^{-(q+1)} x(u+nT)du \Bigg] \\
&= \frac{1}{\Gamma(1-q)} \Bigg[(t+K)^{-q} x(-K+nT) - (t+nT)^{-q} x(0) \\
&\qquad\qquad - q \int_{-nT}^{-K} (t-u)^{-(q+1)} x(u+nT)du \Bigg].
\end{aligned}$$

This yields that

$$I_2 \to \frac{1}{\Gamma(1-q)} \left[(t+K)^{-q} y(-K) - q \int_{-\infty}^{-K} (t-u)^{-(q+1)} y(u)du \right] \tag{21}$$

as $n \to \infty$. We now claim that

$$\begin{aligned}
&\int_{-\infty}^{-K} (t-u)^{-q} y'(u)du \\
&\quad = (t+K)^{-q} y(-K) - q \int_{-\infty}^{-K} (t-u)^{-(q+1)} y(u)du.
\end{aligned} \tag{22}$$

We integrate by parts and use the fact that $\int_{-\infty}^{-K} (t-u)^{-(q+1)} du$ converges to obtain

$$\begin{aligned}
&\int_{-\infty}^{-K} (t-u)^{-q} y'(u)du \\
&= \lim_{b \to \infty} \left[(t-u)^{-q} y(u) \Big|_{-b}^{-K} - q \int_{-b}^{-K} (t-u)^{-(q+1)} y(u)du \right] \\
&= (t+K)^{-q} y(-K) - q \int_{-\infty}^{-K} (t-u)^{-(q+1)} y(u)du.
\end{aligned}$$

This implies that

$$I_2 \to \frac{1}{\Gamma(1-q)} \int_{-\infty}^{-K} (t-u)^{-q} y'(u)du. \tag{23}$$

Combining (20)-(23), we see that ${}^cD^q x(t+nT) \to {}^cD^q_+ y(t)$ as $n \to \infty$. Let $n \to \infty$ in (19), we obtain

$${}^cD^q_+ y(t) = -y(t) + f(t).$$

Thus, $y(t)$ is a T-periodic solution of (16). □

5. Periodic perturbation of forcing

We come now to a sequence of three results based on questions raised in the early stability theory of ordinary differential equations. The reader is referred to the early chapters of the classical book of Bellman [2], for example. We study systems of the form

$$x' = (A + B(t))x$$

with A and B being $n \times n$ matrices and A constant or periodic. The theme is that if $x' = Ax$ is stable in some sense and if $B(t)$ is small or in $L^1[0, \infty)$, then the stability is not disturbed.

Parallel to that study, we now ask similar questions for (1) in the context of asymptotic periodicity. Thus, given an asymptotically stable linear homogeneous equation with constant coefficient:

(i) What is the effect of periodic forcing?

(ii) What is the effect of periodic damping and periodic forcing?

(iii) What is the effect of perturbing the periodicity, itself, in (ii) by the sum of an $L^1[0, \infty)$ function and a function tending to zero in both the damping and the forcing?

The short answer is simple. Because all of the perturbations are convolved with $R(t)$ they damp out to zero and do not affect the long-term behavior at all. To actually prove it is something of a challenge and we introduce some simple but new fixed point ideas. This is a very enlightening property of fractional differential equations.

We have seen that (9) has an asymptotic periodic solution when f is a simple periodic function. Our next result shows that this behavior is unchanged when f is a rather arbitrary function, $h(t, x_t)$, of the history, x_t, of the solution with h periodic in t. To see this, we focus on an equation

$$ {}^cD^q x = -x(t) + h(t, x(t), x(t-r)) =: -x(t) + h(t, x_t), \quad r > 0, \tag{24}$$

for which we seek a solution of the form

$$x(t) = p(t) + q(t) \tag{25}$$

where $p \in \mathcal{P}_T$, the Banach space of continuous T-periodic functions $p : [0, \infty) \to \Re$ with the supremum norm, while $q \in Q$ the Banach space of continuous functions $q : [0, \infty) \to \Re$, $q(t) \to 0$ as $t \to \infty$, with the supremum norm.

Lemma 5.1. *The space $(Y, \|\cdot\|)$ of functions $x = p + q$ with $p \in \mathcal{P}_T$ and $q \in Q$ with the supremum norm is a Banach space.*

See Burton[6, p. 58].

Lemma 5.2. *If $\phi, \eta \in Y$ so is $\phi\eta$. Hence, if $L(x)$ is a polynomial with real coefficients and if $\phi \in Y$, so is $L(\phi)$.*

This is a quick calculation.

Lemma 5.3. *Suppose that $F : \Re \to \Re$ satisfies a local Lipschitz condition. If $x \in Y$, so is $F(x)$.*

Proof. Let $x = p + q$ and let $\|p\| + \|q\| \le K$ for some $K > 0$. Then there exists an $L > 0$ such that

$$|F(p(t) + q(t)) - F(p(t))| \le L|p(t) + q(t) - p(t)| = L|q(t)|.$$

Now both functions on the left are continuous, so the difference defines a continuous function and that function tends to zero since q does. Moreover, $F(p(t))$ is certainly periodic and this proves the result. □

With these definitions and properties, we make the following assumptions for this section. Suppose that for each continuous function $x : [-r, \infty) \to \Re$, the function $h(t, x_t) = h(t, x(t), x(t - r))$ is continuous for $t \ge 0$. Let $\psi : [-r, 0] \to \Re$ be a fixed continuous function and denote its supremum by $\|\psi\|$. For any $\phi \in Y$ with $\phi(0) = \psi(0)$, we define $\phi(s) = \psi(s)$ for $-r \le s \le 0$ (or $\phi_0 = \psi$). Suppose that if $\phi \in Y$ with $\phi_0 = \psi$ then $h(t, \phi_t) \in Y$. Moreover, suppose there is a $K > 0$ such that if $\phi \in Y$ with $\phi_0 = \psi$ and $\|\phi\| \le K$, then $|h(t, \phi_t)| \le K$ whenever $\|\psi\| \le K$. Finally, suppose there is an $\alpha < 1$ such that $\phi, \eta \in Y, \|\phi\|, \|\eta\| \le K$ imply that $|h(t, \phi_t) - h(t, \eta_t)| \le \alpha\|\phi - \eta\|$ for $t \ge 0$ if $\phi_0 = \eta_0 = \psi$ with $\|\psi\| \le K$. If $r = 0$, we simply write $h(t, x_t) = h(t, x(t))$.

Example 5.4. Let

$$^cD^q x = -x + x^2(t) + (1/8)\sin t$$

and take

$$h(t, x(t)) = x^2(t) + (1/8)\sin t, \quad x \in Y, \quad K = 3/8.$$

Then $\|x\| \le K$ implies that $|h(t, x(t))| \le K^2 + (1/8) < K$. Also, if $x, y \in Y$ and $\|x\| \le K, \|y\| \le K$, then

$$|h(t, x(t)) - h(t, y(t))| = |x^2(t) - y^2(t)|$$

$$\le |x(t) - y(t)|(|x(t)| + |y(t)|) \le \|x - y\|(2K) =: \alpha\|x - y\|.$$

Example 5.5. In the same way, let $h(t, x_t) = x(t)x(t-r) + (1/8)\sin t$. If $x, y \in Y, \|x\| \le K, \|y\| \le K$, and if $x_0 = y_0 = \psi$ with $\|\psi\| \le K$, then

$$\begin{aligned}
&|h(t, x_t) - h(t, y_t)| \\
&= |x(t)x(t - r) - y(t)x(t - r) + y(t)x(t - r) - y(t)y(t - r)| \\
&\le |x(t - r)||x(t) - y(t)| + |y(t)||x(t - r) - y(t - r)| \\
&\le 2K\|x - y\|
\end{aligned}$$

so we need $2K < 1$ for a contraction. Also, $\|x\|, \|\psi\| \le K$ yield

$$|h(t, x_t)| \le |x(t)x(t - r)| + (1/8) \le K^2 + (1/8) < K$$

if $1/4 \le K < 1/2$.

We have already seen that $\int_{-\infty}^{0} R(t-s)\phi(s)ds \to 0$ as $t \to \infty$ if $\phi : \Re \to \Re$ is a bounded continuous function.

Theorem 5.6. *Under these conditions, for each continuous initial function $\psi : [-r, 0] \to \Re$,*

$$ {}^{c}D^{q}x = -x + h(t, x_t) \tag{26} $$

has a solution x in Y with $x_0 = \psi$.

Proof. Write the equation as

$$ \begin{aligned} x(t) &= x(0) - \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1}[x(s) - h(s, x_s)]ds \\ &= x(0) - \int_0^t C(t-s)[x(s) - h(s, x_s)]ds \end{aligned} $$

and decompose as

$$ z(t) = x(0) - \int_0^t C(t-s)z(s)ds, $$

$$ z(t) = x(0) - x(0)\int_0^t R(s)ds, $$

$$ x(t) = z(t) + \int_0^t R(t-s)h(s, x_s)ds. $$

For the given $K > 0$, let $\psi : [-r, 0] \to \Re$ be a fixed continuous initial function with $\|\psi\| \leq K$. We define the complete metric space $(Y_K, \|\cdot\|)$ in Y by

$$ Y_K = \{\phi \in Y : \phi(0) = \psi(0), \|\phi\| \leq K\}. $$

Next, recall that for any $\phi \in Y$, we have the extension $\phi_0 = \psi$. Now define $P : Y_K \to Y_K$ by $\phi \in Y_K$ implies that

$$ (P\phi)(t) = z(t) + \int_0^t R(t-s)h(s, \phi_s)ds $$

where

$$ z(t) = \phi(0) - \int_0^t C(t-s)z(s)ds $$

so that

$$ z(t) = \phi(0) - \int_0^t R(t-s)\phi(0)ds = \phi(0)[1 - \int_0^t R(s)ds]. $$

Then

$$ \begin{aligned} |(P\phi)(t)| &\leq K[1 - \int_0^t R(s)ds] + \int_0^t R(t-s)|h(s, \phi_s)|ds \\ &\leq K[1 - \int_0^t R(s)ds] + \int_0^t R(s)dsK = K. \end{aligned} $$

Also, if $\phi, \eta \in Y_K$ then

$$|(P\phi)(t) - (P\eta)(t)| \leq \int_0^t R(t-s)|h(s,\phi_s) - h(s,\eta_s)|ds$$
$$\leq \int_0^t R(t-s)\alpha\|\phi - \eta\| \leq \alpha\|\phi - \eta\|$$

so P is a contraction.

Finally, if $\phi \in Y$ with $\phi_0 = \psi$ so is $h(t,\phi_t) =: p(t) + q(t)$ so

$$\begin{aligned}(P\phi)(t) &= z(t) + \int_0^t R(t-s)h(s,\phi_s)ds \\ &= z(t) + \int_0^t R(t-s)[p(s)+q(s)]ds \\ &= z(t) + \int_0^t R(t-s)q(s)ds - \int_{-\infty}^0 R(t-s)\tilde{p}(s)ds + \int_{-\infty}^t R(t-s)\tilde{p}(s)ds\end{aligned}$$

where $\tilde{p}$ is the T-periodic extension of p on $\Re$. Now $\int_0^t R(t-s)q(s)ds \to 0$ as $t \to \infty$ as it is the convolution of an L^1 function and a function tending to zero. Also

$$\left|\int_{-\infty}^0 R(t-s)\tilde{p}(s)ds\right| \leq \|p\| \int_{-\infty}^0 R(t-s)ds$$
$$= \|p\| \int_t^\infty R(u)du \to 0 \text{ as } t \to \infty.$$

Finally, $\int_{-\infty}^t R(t-s)\tilde{p}(s)ds$ is periodic. Hence, $P : Y_K \to Y_K$ is a contraction with unique fixed point $x \in Y_K$. □

6. A periodically damped equation

Return now to (9) and introduce variable damping in the form

$$^cD^q x(t) = -a(t)x(t) + f(t), \quad 0 < q < 1, \quad x(0) \in \Re, \tag{27}$$

with $a, f : \Re \to \Re$ and both are continuous, while there is a $T > 0$ with

$$a(t+T) = a(t), \quad f(t+T) = f(t).$$

Recall that early on we stated that we always contrive to write our equation as

$$^cD^q x(t) = -[x(t) + G(t, x(t))] + f(t)$$

so that we can decompose the equation and introduce that all-important kernel $R(t-s)$. Clearly, (27) will require a lot of work to achieve this and we suggest that the reader consider the following steps with some care. These steps, and a number of others, were introduced in [9] in an entirely different context and they can be carried out when the $a(t)x(t)$ in (27) is replaced by either a sublinear or a superlinear function.

It is assumed that there are numbers satisfying

$$0 < \epsilon \leq a(t) \leq M. \tag{28}$$

Then we find a positive number $\alpha < 1$ so that for $J = (1/2)(M + \epsilon)$ then

$$|J - a(t)| \leq \alpha J. \tag{29}$$

In fact, we may choose $\alpha = (M - \epsilon)/(M + \epsilon)$.

Theorem 6.1. *If (28) holds, then we have (29) and for every $x(0) \in \Re$ there is a unique solution of (27) in Y. Moreover, for every $x(0)$, the solution converges to the same periodic function as $t \to \infty$.*

Proof. Invert (27) as

$$x(t) = x(0) - \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1}[a(s)x(s) - f(s)]ds$$

and prepare it for separation as

$$\begin{aligned} x(t) &= x(0) - \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1}[Jx(s) - (J - a(s))x(s) - f(s)] \\ &= x(0) - \frac{J}{\Gamma(q)} \int_0^t (t-s)^{q-1}[x(s) - \frac{J - a(s)}{J}x(s) - \frac{f(s)}{J}]ds. \qquad (30) \end{aligned}$$

The new kernel $\frac{J}{\Gamma(q)}t^{q-1}$ is still completely monotone and there is a resolvent, say $R(t)$ again, with exactly the same properties as before. It is crucial that J be positive. Thus, we decompose (30) as

$$z(t) = x(0)[1 - \int_0^t R(s)ds]$$

and

$$x(t) = z(t) + \int_0^t R(t-s)\left[\frac{J - a(s)}{J}x(s) + \frac{f(s)}{J}\right]ds. \tag{31}$$

Because of (29) this equation will define, for the given $x(0)$, a contraction mapping on the Banach space of bounded continuous functions $\phi : [0, \infty) \to \Re$ with the supremum norm and there is a unique bounded solution for every $x(0) \in \Re$. More to the point here, we can define the same mapping of the space $(Y, \|\cdot\|)$ into itself and there is a fixed point in Y. The point is that if $x = p + q \in Y$ then

$$\frac{J - a(s)}{J}x(s) = p^*(s) + q^*(s) \in Y$$

and

$$\int_0^t R(t-s)[p^*(s) + q^*(s)]ds \in Y.$$

Moreover, if $x_1(0)$ and $x_2(0)$ are given points and if $z_1(t)$, $z_2(t)$ and $x_1(t)$, $x_2(t)$ are the corresponding solutions then $x_1(t) - x_2(t)$ solves

$$x(t) = z_1(t) - z_2(t) + \int_0^t R(t-s)\frac{J - a(s)}{J}x(s)ds.$$

This equation will define a contraction mapping on the Banach space of bounded continuous functions tending to zero, showing that the unique solution tends to zero. ☐

7. Perturbation of the periodicity, itself

Previously, we perturbed damping and forcing with periodic functions and we saw that the resulting solution remained asymptotically periodic. Now our perturbations are perturbations of the periodicity itself. Consider a fractional differential equation of Caputo type

$$ {}^cD^qx = -[a(t)+b(t)+c(t)]x(t)+[f(t)+g(t)+h(t)],\ 0<q<1, \tag{32}$$

with $a, b, c, g, f, h : [0, \infty) \to \Re$ all continuous,

$$(i) \quad b(t), h(t) \to 0 \quad \text{as } t \to \infty,$$

$$(ii) \quad c(t), g(t) \in L^1[0, \infty),$$

$$(iii) \quad f(t), g(t) \quad \text{bounded},$$

and there are positive numbers ϵ, M with

$$(iv) \quad \epsilon \le a(t) \le M.$$

Theorem 7.1. *Equation (32) has a bounded solution for each $x(0) \in \Re$ and the solution is unique. If $f(t) \equiv 0$, then that solution converges to zero. If $x_1(t), x_2(t)$ are solutions for different initial conditions, then $x_1(t) - x_2(t) \to 0$ as $t \to \infty$.*

Theorem 7.2. *If $a(t+T) = a(t)$ and $f(t+T) = f(t)$ for all t and some $T > 0$, then that unique solution of (32) lies in the space $(Y, \|\cdot\|)$ defined in Lemma 5.1.*

In both of these theorems the uniqueness follows from a Lipschitz condition and not from the fixed point theorem that will be used for the rest of the conclusion.

Invert (32) as

$$x(t) = x(0) - \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\{[a(s)+b(s)+c(s)]x(s)-[f(s)+g(s)+h(s)]\}ds. \tag{33}$$

As in the last section, we can find $J > 1$ and $\alpha < 1$ with $|a(t)-J| \le \alpha J$; a first step is $J = \epsilon + (1/2)[M - \epsilon]$. Then rewrite the equation as

$$\begin{aligned} x(t) = x(0) - \frac{J}{\Gamma(q)}\int_0^t (t-s)^{q-1}\{x(s) - \frac{(J-a(s))}{J}x(s) + \frac{(b(s)+c(s))}{J}x(s) \\ - \frac{1}{J}[f(s)+g(s)+h(s)]\}ds. \end{aligned} \tag{34}$$

Decompose into

$$z(t) = x(0)[1 - \int_0^t R(s)ds]$$

and

$$\begin{aligned}
x(t) &= z(t) + \frac{1}{J}\int_0^t R(t-s)[f(s)+g(s)+h(s)]ds \\
&+ \int_0^t R(t-s)\{\frac{(J-a(s))}{J}x(s)\}ds - \frac{1}{J}\int_0^t R(t-s)(b(s)+c(s))x(s)ds \\
&=: z(t) + F(t) + G(t) + H(t) + \int_0^t R(t-s)\{\frac{(J-a(s))}{J}x(s)\}ds \\
&- \frac{1}{J}\int_0^t R(t-s)(b(s)+c(s))x(s)ds \\
&=: (Bx)(t) \\
&+ (Ax)(t).
\end{aligned} \tag{35}$$

The notation should clearly indicate that we have defined functions F, G, H, together with operators B and A. Each of these will have properties which were clearly enhanced by the kernel R. Note that $R(t) \to 0$ as $t \to \infty$ and $R \in L^1[0,\infty)$. As $h(t) \to 0$, we have $H(t) \to 0$ as $t \to \infty$. The property is also true for G. The function F is bounded.

There are two important, but simple, properties.

Theorem 7.3. *Let $(X, \|\cdot\|)$ denote the Banach space of bounded continuous functions $\phi : [0,\infty) \to \Re$ with the supremum norm. Then $B : X \to X$ is a contraction with unique fixed point.*

Theorem 7.4. *Let $(X_0, \|\cdot\|)$ be the subspace of $(X, \|\cdot\|)$ with $\phi(t) \to 0$ as $t \to \infty$. If $F(t) = 0$, then for B in (35) it follows that $B : X_0 \to X_0$ is a contraction with unique fixed point.*

The next result is not so simple, but a proof is found in [9](also see Lemma 7.7 below).

Theorem 7.5. *The operator A in (35) maps bounded subsets of X into equicontinuous subsets of X.*

We must modify this in order to prove that $B + A$ has a fixed point because $c(t)$ can be unbounded, but the modification is simple.

It is a useful fact that if $f(t+T) = f(t)$, then $F \in Y$; that is,

$$F(t) = p(t) + q(t), \quad p(t+T) = p(t), \quad q(t) \to 0 \quad \text{as } t \to \infty.$$

It turns out that $A + B$ is almost a contraction. It fails because $\int_0^t R(t-s)(|b(s)| + |c(s)|)ds$ can be large on a certain interval which we will denote by $[0, S]$; but for $t > S$, then $A + B$ is a contraction. We readily avoid the difficulty by using the following result from [8].

Theorem 7.6. *Let $(X, \|\cdot\|)$ be a Banach space, $A, B : X \to X$, B a contraction with constant $\alpha < 1$, and A continuous with A mapping bounded sets into compact sets. Either*

(i) $x = \lambda B(x/\lambda) + \lambda Ax$ *has a solution in* X *for* $\lambda = 1$, *or*
(ii) *the set of all such solutions,* $0 < \lambda < 1$, *is unbounded.*

Notice that, as B linear, the λs cancel in that first term.

We are now ready for the proof of Theorem 7.1. It will be proved by a sequence of lemmas allowing us to apply Theorem 7.6.

Lemma 7.7. *Let Z be any bounded subset of X. Then AZ resides in a compact subset of X.*

Proof. Let the bound on Z be L and let $\psi(t) := \frac{1}{J}\int_0^t R(t-s)|b(s) + c(s)|ds$ so that if $\phi \in Z$ then

$$|(A\phi)(t)| \leq L\psi(t) \to 0 \quad \text{as } t \to \infty. \tag{36}$$

There are two steps for the equicontinuity. Let $\epsilon > 0$ be given and find T so that $L\psi(t) < \epsilon/2$ if $t \geq T$. Thus, if $\phi \in Z$ and if $t_1, t_2 \geq T$ then $|(A\phi)(t_1) - (A\phi)(t_2)| < \epsilon$. Next, with T fixed, we consider AZ for $0 \leq t \leq T$ and invoke the result from [9] to conclude that this set is equicontinuous. Hence, we see that AZ, itself, is equicontinuous, and it is contained in a compact subset of X since (36) holds. This now completes the proof. □

We have already noted that B is a contraction so all that remains in order to invoke Theorem 7.6 is to show that there is an *a priori* bound on solutions of

$$x(t) = (Bx)(t) + \lambda(Ax)(t), \quad 0 < \lambda < 1. \tag{37}$$

Lemma 7.8. *There is a $K > 0$ so that if x solves (37) on $[0, \infty)$ for any such λ, then $\|x\| \leq K$.*

Proof. For the given α with $|J - a(t)| < \alpha J$, refer to ψ in the proof of Lemma 7.7 and find a number S such that $\psi(t) \leq (1-\alpha)/2$ for $t \geq S$. Next, find a number D with $|z(t) + F(t) + G(t) + H(t)| \leq D$. We will now find a bound on an arbitrary solution x of (37). Suppose, by way of contradiction, that for some such x there is a sequence $t_n \to +\infty$ with $|x(t)| \leq |x(t_n)|$ if $0 \leq t \leq t_n$ and $|x(t_n)| \uparrow +\infty$. Then for $t_n > S$ we have

$$|x(t_n)| \leq D + |x(t_n)| \int_0^{t_n} R(t_n - s)\left[\alpha + \frac{|b(s) + c(s)|}{J}\right] ds \leq D + |x(t_n)|(\alpha + \psi(t_n))$$

and this will contradict $|x(t_n)| \uparrow \infty$ and yield a bound, say U, if any of these t_n lie to the right of S. Obviously, the t_n would depend on the particular solution, but the bound would not so depend if any of these t_n lie to the right of S, which is independent of the particular x.

Thus, we suppose that all the $t_n \leq S$. Let $Q \geq |b(s)| + |c(s)|$ for $0 \leq s \leq S$. Then there is a Gronwall inequality [10, p. 111] (consult also [13, p. 188], [10, pp. 172, 173], [21] [15, p. 91]) of the following form:

$$|x(t)| \le D + \int_0^t R(t-s)Q|x(s)|ds \le D + \frac{J}{\Gamma(q)}\int_0^t (t-s)^{q-1}Q|x(s)|ds$$

implies

$$|x(t)| \le DE_q(JQt^q) \text{ for } 0 \le t \le S$$

where E_q is the Mittag-Leffler function of order q. This function is continuous on $[0, S]$ and, hence, has a bound, say N. The bound is uniform for all the values of λ and all such x. Either U or N is a bound on $[0, \infty)$. □

We now apply Theorem 7.6 to prove the first conclusion in Theorem 7.1. Continue in the same way with the subspace $(X_0, \|\cdot\|)$ when $f(t) \equiv 0$ to prove the second conclusion in Theorem 7.1. Finally, the difference of two solutions satisfies an equation with $f(t)$ removed so it tends to zero.

Now we prove Theorem 7.2. Note that $z(t), H(t), G(t), \int_0^t R(t-s)b(s)ds \to 0$ as $t \to \infty$, while $\alpha(t) := \frac{J-a(t)}{J}$ and $F(t)$ lie in space Y of Lemma 5.1. Also, if P is the operator defined by $\phi \in Y$ implies that

$$(P\phi)(t) = (B\phi)(t) + \lambda(A\phi)(t)$$

then $P\phi \in Y$. To see this, notice first that

$$\begin{aligned}
&\int_0^t R(t-s)\alpha(s)[p(s)+q(s)]ds \\
&= \int_{-\infty}^t R(t-s)\alpha(s)\tilde{p}(s)ds + \left[\int_0^t R(t-s)\alpha(s)q(s)ds - \int_{-\infty}^0 R(t-s)\alpha(s)\tilde{p}(s)ds\right] \\
&=: p_1(t) + q_1(t) \in Y
\end{aligned}$$

where $\tilde{p}$ is again the T-periodic extension of p on $\Re$. Also, $A(p+q) \in Y$.

Theorem 7.6 now says that there is a solution in Y.

Theorem 7.9. *Let the conditions of Theorem 6.1 and 7.2 hold. If $p_1(t)$ is the periodic solution to which all solutions of (27) converge and if $p_2(t)$ is the periodic function to which all solutions of (32) converge, then $p_1(t) = p_2(t)$.*

Proof. In both Theorem 6.1 and 7.2 we have $F(t) = \frac{1}{J}\int_0^t R(t-s)f(s)ds$ where f is defined in (27). Write (31) as

$$y(t) = z(t) + F(t) + \int_0^t R(t-s)\alpha(s)y(s)ds$$

and $x(t)$ the solution of (32) as

$$x(t) = z(t)+F(t)+G(t)+H(t)+\int_0^t R(t-s)[\alpha(s)-\frac{1}{J}(b(s)+c(s))]x(s)ds$$

where both have the same initial condition so $z(t)$ is the same in each case. Now $x(t)$ is a bounded function and so $\frac{1}{J}\int_0^t R(t-s)[-b(s)-c(s)]x(s)ds =: \xi(t) \to 0$ as $t \to \infty$. We subtract y from x obtaining

$$x(t) - y(t) = H(t) + G(t) + \int_0^t R(t-s)[\alpha(s)][x(s) - y(s)]ds + \xi(t).$$

Then define $L(t) = H(t) + G(t) + \xi(t) \to 0$ as $t \to \infty$ and write that last equation as

$$w(t) = L(t) + \int_0^t R(t-s)\alpha(s)w(s)ds.$$

That last equation defines a contraction mapping on $(X_0, \|\cdot\|)$ with solution tending to zero. But

$$x(t) = p_1(t) + q_1(t) \in Y, \quad y(t) = p_2(t) + q_2(t) \in Y$$

while

$$w(t) = (p_1(t) - p_2(t)) + (q_1(t) - q_2(t)) \to (p_1(t) - p_2(t)) = 0,$$

as required. □

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