ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF FUNCTIONAL DIFFERENTIAL EQUATIONS BY FIXED POINT THEOREMS

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ABSTRACT. In a series of papers we have studied stability properties of functional differential equations by means of fixed point theory. We enlarge that study now by also considering delay equations which may be unstable when the delay is zero. We continue to focus on challenging examples to illustrate the work, as opposed to attempting to state general theorems. Here, we deal with three different examples. In Part I we obtain asymptotic stability using Schauder's and Banach's fixed point theorems; it advances results used by Liapunov techniques in several ways, but particularly by placing no smoothness on the delay. It also advances some of our earlier work with fixed point theory in that it places no smoothness conditions on the nonlinear perturbation. In Part II we prove boundedness and asymptotic stability using Krasnoselskii's fixed point theorem. Schaefer's fixed point theorem is used to prove that there is a periodic solution when a periodic forcing function is added to that equation.

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1. OVERVIEW

Liapunov's direct method is by far the most general method for dealing with stability, boundedness, and the existence of periodic solution of ordinary and functional differential equations. But over the last hundred years investigators have encountered a number of difficulties which have remained unresolved. Several years ago we began a project [2-5] to see if those problems could be circumvented using fixed point theory. Since all of those problems had originated with specific examples, it seemed that our endeavor should center around difficult examples. Thus, we have offered no general theorems. Instead, we have presented examples which are known to cause severe difficulties using Liapunov's direct method.

Our results here are of two different types. In Part I we look at a typical problem on which there has been much work using Liapunov functionals and we show how fixed point theory can be used to circumvent several difficulties encountered in that method. In Part II we continue studying that problem under conditions which have prevented the use of Liapunov functions; we show that fixed point theory can be used to show boundedness, asymptotic stability, and the existence of periodic solutions.

Here is a general overview of the work. The equation

$$x'(t) = -a(t)x(t - r_1) + b(t)g(x(t - r_2(t)))$$

has been studied by means of several Liapunov functionals under different assumptions. In most, but not all, cases it is assumed that

(A) $r_1 = 0$

and

(B) $|g(x)| \leq |x|$ for small x.

If, in addition, one asks that

(C) r_2 is a positive constant,

then one is led to the condition

(D) $a(t) \geq |b(t + r_2)|$

and to the Liapunov functional

$$V(t, x_t) = |x(t)| + \int_{t-r_2}^{t} |b(s + r_2)||g(x(s)|ds$$

with derivative satisfying

$$V'(t, x_t) \leq -a(t)|x(t)| + |b(t)g(x(t - r_2))| + |b(t + r_2)g(x(t))| - |b(t)g(x(t - r_2))| \leq 0.$$

We have V positive definite and $V' \leq 0$ so the zero solution is stable. With additional conditions asymptotic stability can be obtained.

Investigators have successfully removed (C) at some cost. We can ask instead of (C) that there are positive constants α and k with

(E) $r_2(t)$ continuous and positive, $1 - r_2'(t) \geq \alpha$, and

(F) $a(t) \geq k \geq |b(t)|/\alpha$

which leads to the Liapunov functional

$$V(t, x_t) = |x(t)| + k \int_{t-r_2(t)}^{t} |g(x(s))| ds$$

whose derivative can be made non-positive.

Our project seeks to reduce or remove (A)-(F) by means of fixed point theory.

In an earlier paper [3] we used contraction mappings by assuming (A) and (B), but eliminating (C) and (E), while reducing (D) to an averaging condition, at the cost of asking

(G) that g satisfies a local Lipschitz condition.

Part I of this paper uses Schauder's fixed point theorem to remove (G), resulting in conditions close to (A), (B), and an averaging form of (D) implying asymptotic stability. This is a significantly better result than has been obtained via Liapunov functionals.

In Part II we continue with the equation, but we violate every one of the conditions (A) - (G) and consider

$$x'(t) = -a(t)x(t - r_1) + b(t)x^{1/3}(t - r_2(t))$$

where r_1 is a positive constant. Notice now that $g(x) = x^{1/3}$ and so x does not dominate g. If $r_i = 0$ then it is a Bernoulli equation; with $a(t) > 0$ and $b(t) < 0$ a simple Liapunov function will at least yield stability and sometimes asymptotic stability. On the other hand, if a and b are positive constants then the zero solution can be non-unique to the right, and hence, there can be no stability. But, surprisingly, if $a(t) > 0$ and dominates $b(t)$ sufficiently strongly then the zero solution can be asymptotically stable independent of the sign of $b(t)$. Thus, this equation is very rich in behavior of solutions.

When the r_i are positive, then it is a challenging problem and provides a good test for the effectiveness of fixed point theory as a tool for proving stability and boundedness. Our first result uses Krasnoselskii's fixed point theorem to prove boundedness of solutions in the case corresponding to a and b being positive constants mentioned above. Our third result yields asymptotic stability via Krasnoselskii's fixed point theorem when a dominates b sufficiently strongly.

For our second result we suppose that our equation is given a periodic perturbation and that a, b are also periodic. Then, under essentially the same conditions as our first result, we obtain a periodic solution by means of Schaefer's fixed point theorem. This result is again related to Liapunov theory. It is well known that if a system of differential equations has a globally Lipschitz Liapunov function with negative definite derivative, then the equation can be given a sufficiently small perturbation and that same Liapunov function will yield boundedness. Our second result offers a parallel property.

It is to be noted that other authors are also using fixed point theory to prove stability, although in different contexts. For example, Serban [10] has used a Picard operator to prove asymptotic stability of a difference equation. Rus [8] uses Picard operators to prove data dependence of boundary value problems.

PART I

Asymptotic Stability: Two fixed point theorems

2. PRELIMINARIES

In this part, we discuss asymptotic behavior of solutions of functional differential equations by using the following theorem, which can be found in Smart [5; p. 15]. For reference it may be stated as follows.

THEOREM 2.1 (Schauder's first theorem). Let $(C, \|\cdot\|)$ be a normed space, and let S be a compact convex nonempty subset of C. Then every continuous mapping of S into S has a fixed point.

First we introduce a concept of equi-convergence. Let r_0 be a fixed nonnegative constant.

DEFINITION (Equi-convergence). The set S of real-valued functions on $[-r_0, \infty)$ is said to be equi-convergent to 0 if there is a function $q : R^+ \to R^+ := [0, \infty)$ such that $q(t) \to 0$ as $t \to \infty$, and that for any $\phi \in S$ and $t \in R^+$, $|\phi(t)| \le q(t)$ holds.

Now we prepare an Ascoli-Arzela like lemma. For the proof, see Lemma 3 in Burton and Furumochi [4].

LEMMA. If the set $\{\phi_k(t)\}$ of real-valued functions on $[-r_0, \infty)$ is equi-convergent to 0 with respect to a function $q : R^+ \to R^+$, and is equi-continuous, then the sequence $\{\phi_k(t)\}$ contains a subsequence that converges uniformly on R^+ to a continuous function $\phi(t)$ with $|\phi(t)| \le q(t)$ on R^+.

3. FIRST ORDER HALF-LINEAR EQUATIONS

First consider the scalar half-linear equation

$$x'(t) = -a(t)x(t) - b(t)g(x(t - r(t))), t \in R^+, \tag{3.1}$$

where $a, b, r : R^+ \to R := (-\infty, \infty)$ and $g : R \to R$ are continuous. Let α be any fixed positive number. We assume that there are constants $\beta > 0, \gamma > 0$ and $r_0 \geq 0$ so that

$$|g(x)| \leq \beta|x| \text{ for } |x| \leq \alpha, \tag{3.2}$$

$$\sup\{e^{\int_\tau^t (a(s) - \beta\gamma|b(s)|)ds} | t \in R^+\} \leq \gamma, \tag{3.3}$$

where $\tau = \tau(t) := \max(0, t - r(t))$,

$$\int_0^t (a(s) - \beta\gamma|b(s)|)ds \to \infty \text{ as } t \to \infty, \tag{3.4}$$

and

$$r(t) \geq 0 \text{ and } t - r(t) \geq -r_0. \tag{3.5}$$

Define a number δ by

$$\delta := \alpha e^{-\sigma}, \tag{3.6}$$

where $\sigma := \sup\{\int_0^t (\beta\gamma|b(s)| - a(s))ds | t \in R^+\}$.

Now let $q : [-r_0, \infty) \to R$ be a continuous function such that $q(t) = q(0)$ on $[-r_0, 0]$, and that $q(t)$ is the unique solution of the initial value problem

$$q' = (\beta\gamma|b(t)| - a(t))q, \ q(0) = \delta, \ t \in R^+.$$

Then $q(t)$ can be expressed as

$$q(t) = \delta e^{-\int_0^t a(s)ds} + \beta\gamma \int_0^t e^{-\int_s^t a(u)du} |b(s)| q(s) ds$$

$$= \delta e^{\int_0^t (\beta\gamma|b(s)| - a(s))ds}, \ t \in R^+,$$

which together with (3.4) and (3.6), implies

$$q(t) \to 0 \text{ as } t \to \infty \tag{3.7}$$

and

$$0 < q(t) \leq \delta e^\sigma = \alpha, t \in R^+. \tag{3.8}$$

Now we have the following theorem which effectively remedies the deficiencies we had listed for the Liapunov functional.

THEOREM 3.1. Suppose that solutions of (3.1) are uniquely determined by continuous initial functions. If (3.2)-(3.5) hold, then the solutions of Equation (3.1) with small initial functions tend to 0 as $t \to \infty$.

Proof. We will give the details for the initial time $t_0 = 0$, but general initial time t_0 offers no new difficulties. Let C be the Banach space of bounded and continuous functions $\phi : [-r_0, \infty) \to R$ with the supremum norm $\|\cdot\|$. For a continuous function $\psi : [-r_0, 0] \to R$ with $\sup\{|\psi(\theta)|| - r_0 \le \theta \le 0\} \le \delta$, let S be a set of continuous functions $\phi : [-r_0, \infty) \to R$ such that $\phi(t) = \psi(t)$ on $[-r_0, 0]$, $|\phi(t)| \le q(t)$ on R^+, and $|\phi(t_1) - \phi(t_2)| \le L|t_1 - t_2|$ for $t_1, t_2 \in R^+$ with $\tau_1 \le t_1, t_2 \le \tau_2$ where $L = L(\tau_1, \tau_2)$ is a function with

$$(|a(t))| + \beta\gamma|b(t)|)\alpha \le L \text{ for } \tau_1 \le t \le \tau_2.$$

Since we have (3.8), we obtain

$$|q'(t)| \le (|a(t)| + \beta\gamma|b(t)|)\alpha, t \in R^+.$$

Thus the function $\xi(t)$ defined by $\xi(t) = \psi(t)$ on $[-r_0, 0]$, and $\xi(t) = \psi(0)q(t)/\delta$ on R^+ is an element of S, and from (3.7) and Lemma, S is a compact convex nonempty subset of C. Define a mapping P for $\phi \in S$ by

$$(P\phi)(t) = \psi(t) \text{ if } -r_0 \le t \le 0,$$

and

$$(P\phi)(t) = \psi(0)e^{-\int_0^t a(s)ds} + \int_0^t e^{-\int_s^t a(u)du} b(s)g(\phi(s - r(s)))ds \text{ if } t > 0.$$

Then we have $(P\phi)(t) = \psi(t)$ on $[-r_0, 0]$, and from (3.2) we obtain

$$|(P\phi)(t)| \le \delta e^{-\int_0^t a(s)ds} + \beta \int_0^t e^{-\int_s^t a(u)du}|b(s)|q(s - r(s))ds$$

$$\le \delta e^{-\int_0^t a(s)ds} + \beta\gamma \int_0^t e^{-\int_s^t a(u)du}|b(s)|q(s)ds = q(t), t \in R^+.$$

Moreover, it is easy to see that

$$(P\phi)'(t) = -a(t)(P\phi)(t) + b(t)g(\phi(t - r(t))), t > 0,$$

which implies

$$|(P\phi)'(t)| \le |a(t)|q(t) + \beta|b(t)|q(t - r(t))$$

$$\le (|a(t)| + \beta\gamma|b(t)|)q(t) \le (|a(t)| + \beta\gamma|b(t)|)\alpha, t > 0,$$

and hence, P maps S into S. Clearly P is continuous. Thus, by Theorem 2.1, P has a fixed point ϕ and that is the solution which satisfies $|\phi(t)| \le q(t)$, $t \in R^+$.

Now we show an example.

EXAMPLE 3.1. Let $a : R^+ \to R$ be a 13-periodic function satisfying

$$a(t) = \begin{cases} -1, & 0 \le t < 1 \\ 6t - 7, & 1 \le t < 2 \\ 5, & 2 \le t < 12 \\ 77 - 6t, & 12 \le t \le 13, \end{cases}$$

and let $r(t) \equiv r$, $b(t) \equiv 2, \beta = 1$ and $0 \le r \le \ln 2$. Then it is easily seen that (3.3) and (3.4) hold with $\gamma = 2$, $\sigma = 85/12$ in (3.6), and $q(t) = \delta e^{\int_0^t (4-a(s))ds} \to 0$ as $t \to \infty$, where δ is defined by (3.6). Thus, by Theorem 3.1, for a continuous function $\psi : [-r, 0] \to R$ with $\sup\{|\psi(\theta)|| - r \le \theta \le 0\} \le \delta$, the solution $x(t, 0, \psi)$ of the equation

$$x'(t) = -a(t)x(t) - b(t)g(x(t-r)), t \in R^+,$$

satisfies

$$|x(t, 0, \psi)| \le \delta e^{\int_0^t (4-a(s))ds}, t \in R^+.$$

If $g(x) \equiv x$ on R and $r(t) \equiv r$, Equation (3.1) becomes

$$x'(t) = -a(t)x(t) - b(t)x(t-r), t \in R^+. \tag{3.9}$$

In Hale [6; p. 108], under the assumption

$$a(t) \ge \delta > 0, |b(t)| \le \theta\delta, \theta < 1, \tag{3.10}$$

where δ and θ are constants, the uniform asymptotic stability of the zero solution of Equation (3.9) is discussed by using the Liapunov functional

$$V(t, x_t) = (1/2)(x^2(t) + \delta \int_{t-r}^{t} x^2(s)ds).$$

On the other hand, in Burton and Furumochi [3], under the assumption

$$\int_0^t e^{-\int_s^t a(u)du}|b(s)|ds \le \eta < 1 \text{ on } R^+, \int_0^t a(s)ds \to \infty \text{ as } t \to \infty, \tag{3.11}$$

where η is a constant, the asymptotic stability of the zero solution of Equation (3.9) is discussed by using the contraction mapping principle.

But the functions $a(t)$ and $b(t)$ in Example 3.1 satisfy neither (3.10) nor (3.11).

Next consider the scalar integro-differential equation

$$x'(t) = -a(t)x(t) + \int_{t-r(t)}^{t} b(t, s)g(x(s))ds, t \in R^+, \tag{3.12}$$

where $a, r : R^+ \to R, b : R^+ \times R \to R$ and $g : R \to R$ are continuous. Let α be any fixed number. We assume that there are constants $\beta > 0, \gamma > 0$ and $r_0 \geq 0$ so that (3.2), (3.5),

$$\sup\{\sup\{e^{\int_v^t (a(s) - \beta\gamma \int_{s-r(s)}^s |b(s,u)|du)ds} | \tau \leq v \leq t\} | t \in R^+\} \leq \gamma, \tag{3.13}$$

where $\tau = \tau(t) := \max(0, t - r(t))$, and

$$\int_0^t (a(s) - \beta\gamma \int_{s-r(s)}^s |b(s,u)|du)ds \to \infty \text{ as } t \to \infty. \tag{3.14}$$

For the number $\sigma := \sup\{\int_0^t (\beta\gamma \int_{s-r(s)}^s |b(s,u)|du - a(s))ds | t \in R^+\}$, define a number δ by $\delta := \alpha e^{-\sigma}$.

Now let $q : [-r_0, \infty) \to R$ be a continuous function such that $q(t) \equiv q(0)$ on $[-r_0, 0]$, and that $q(t)$ is the unique solution of the initial value problem

$$q' = (\beta\gamma \int_{t-r(t)}^t |b(t,s)|ds - a(t))q, \ q(0) = \delta, \ t \in R^+.$$

Then $q(t)$ can be expressed as

$$\begin{aligned} q(t) =& \delta e^{-\int_0^t a(s)ds} + \beta\gamma \int_0^t e^{-\int_s^t a(u)du} \int_{s-r(s)}^s |b(s,u)|du q(s)ds \\ =& \delta e^{\int_0^t (\beta\gamma \int_{s-r(s)}^s |b(s,u)|du - a(s))ds}, \ t \in R^+, \end{aligned}$$

which together with (3.14) and $\delta = \alpha e^{-\sigma}$, implies (3.7) and (3.8).

THEOREM 3.2. Suppose that solutions of (3.12) are uniquely determined by continuous initial functions. If (3.2), (3.5), (3.13) and (3.14) hold, then the solutions of Equation (3.12) with small initial functions tend to 0 as $t \to \infty$.

This theorem can be easily proved by taking the set S in the proof of Theorem 3.1 for the above function $q(t)$ and a function $L = L(\tau_1, \tau_2)$ with

$$(a(t) + \beta\gamma \int_{t-r(t)}^t |b(t,s)|ds)\alpha \leq L \text{ for } \tau_1 \leq t \leq \tau_2,$$

and by defining a mapping P for $\phi \in S$ by

$$(P\phi)(t) := \begin{cases} \psi(t), & -r_0 \leq t \leq 0, \\ \psi(0)e^{-\int_0^t a(s)ds} + \int_0^t e^{-\int_s^t a(u)du} \int_{s-r(s)}^s b(s,u)du g(\phi(s))duds, & t > 0. \end{cases}$$

So we omit the details of the proof.

Now we show an example.

EXAMPLE 3.2. Let $a : R^+ \to R$ be the function defined in Example 3.1, and let $r(t) \equiv r$, $b(t,s) \equiv 2$, $\beta = 1$ and $0 \le r \le 2/3$. Then it is easily seen that (3.13) and (3.14) hold with $\gamma = 3$, $\sigma \le 85/12$ and $q(t) = \delta e^{\int_0^t (6r - a(s)ds)} \to 0$ as $t \to \infty$, where $\delta := \alpha e^{-\sigma}$. Thus, by Theorem 3.2, for a continuous function $\psi : [-r, 0] \to R$ with $\sup\{|\psi(\theta)| - r \le \theta \le 0\} \le \delta$, the solution $x(t, 0, \psi)$ of the equation

$$x'(t) = -a(t)x(t) + 2\int_{t-r}^{t} g(x(s))ds, \ t \in R^+,$$

satisfies

$$|x(t, 0, \psi)| \le \delta e^{\int_0^t (6r - a(s))ds}, \ t \in R^+.$$

In Burton and Furumochi [3], under the assumption

$$\text{there exists } \eta < 1 \text{ with } \int_0^t e^{-\int_s^t a(u)du} \int_{s-r(s)}^{s} |b(s,u)| du ds \le \eta, \tag{3.15}$$

the asymptotic stability of the zero solution of Equation (3.12) is discussed by using the contraction mapping principle. But the functions $a(t)$ and $b(t,s)$ in Example 3.2 do not satisfy (3.15) for $r = 2/3$.

4. A SECOND ORDER EQUATION

All equations in the previous section are first order. We now discuss stability of solutions of a second order equation. Here, the reader will see that problems will occur in the variation of parameters formula if we try to do this with variable coefficients in the part of the equation from which we hope to derive stability. But there are ways to choose the system from the second order equation so that the variable matrix $A(t)$ will commute with its integral. In that case, the work can proceed in the manner given below.

We discuss asymptotic behavior of solutions using the contraction mapping principle. Consider the linear equation of retarded type

$$x'' + ax' + bx(t-r) = 0 \tag{4.1}$$

and write it as the system

$$\begin{aligned} x' &= y \\ y' &= -\, ay - bx + (d/dt)\int_{t-r}^{t} bx(s)ds \end{aligned} \tag{4.2}$$

which is then expressed as the vector system

$$z' = Az + (d/dt)\int_{t-r}^{t} Bz(s)ds, \tag{4.3}$$

where A and B are

$$A = \begin{pmatrix} 0 & 1 \\ -b & -a \end{pmatrix}$$

and

$$B = \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix}.$$

By the variation of parameters formula

$$z(t) = e^{At} z_0 + \int_0^t e^{A(t-s)} (d/ds) \int_{s-r}^{s} Bz(u)duds \tag{4.4}$$

or upon integration by parts,

$$\begin{aligned} z(t) =& e^{At} z_0 + \int_{t-r}^{t} Bz(u)du - e^{At} \int_{-r}^{0} Bz(u)du \\ &+ A \int_0^t e^{A(t-s)} \int_{s-r}^{s} Bz(u)duds. \end{aligned} \tag{4.5}$$

In order for $e^{At} \to 0$ as $t \to \infty$ we need

$$a > 0, \; b > 0. \tag{4.6}$$

Let $\psi : [-r, 0] \to R^2$ be a given continuous initial function and define

$$S := \{\phi : [-r, \infty) \to R^2 | \text{ continuous }, \phi_0 = \psi, \phi(t) \to 0 \text{ as } t \to \infty\},$$

where ϕ_0 is defined by $\phi_0(\theta) := \phi(\theta)$ for $-r \leq \theta \leq 0$. Then define a mapping P for $\phi \in S$ by

$$\begin{aligned} (P\phi)(t) :=& e^{At} \psi(0) + \int_{t-r}^{t} B\phi(u)du - e^{At} \int_{-r}^{0} B\psi(u)du \\ &+ A \int_0^t e^{A(t-s)} \int_{s-r}^{s} B\phi(u)duds. \end{aligned} \tag{4.7}$$

Here we use special norms for a vector and a matrix. For

$$z = \begin{pmatrix} x \\ y \end{pmatrix}$$

let $|z|_0 := |x| + |y|$. Let Q be a fixed 2×2 nonsingular matrix such that $|q|_0 \leq 1$, where q is the second column of Q, and let $|z| := |Qz|_0$. For a 2×2 matrix M, let

$$|M| := \sup\{|QMQ^{-1}z|_0 : |z|_0 = 1\}.$$

Then $|M|$ is the norm of M.

Now we have the following theorem.

THEOREM 4.1. Let $a > 0$ and $b > 0$. In addition, if

$$br(1 + \int_0^t |Ae^{A(t-s)}|ds) < 1 \tag{4.8}$$

holds, then every solution of Equation (4.1) tends to 0 with its derivative as $t \to \infty$.

Since e^{At} is an L^1-function on R^+, if $\phi \in S$ then $(P\phi)(t) \to 0$ as $t \to \infty$. Thus P maps S into S, and it is easily seen that (4.8) implies that P is a contraction mapping. So we omit the details of the proof.

Finally we show an example.

EXAMPLE 4.1. Consider the equation

$$x'' + \frac{1}{3}x' + \frac{1}{48}x(t-16) = 0, t \in R^+. \tag{4.9}$$

Then

$$A = \begin{pmatrix} 0 & 1 \\ -1/48 & -1/3 \end{pmatrix}$$

in Equation (4.3), and the characteristic values of A are $-1/4$ and $-1/12$. Let Q be a 2×2 nonsingular matrix such that

$$Q = \begin{pmatrix} -\frac{1}{16} & -\frac{3}{4} \\ \frac{1}{16} & \frac{1}{4} \end{pmatrix}.$$

Then we have

$$QAQ^{-1} = \begin{pmatrix} -1/4 & 0 \\ 0 & -1/12 \end{pmatrix}.$$

It is easy to see that $Ae^{A(t-s)} = Q^{-1}EQ$, where

$$E := \begin{pmatrix} -(1/4)e^{-(t-s)/4} & 0 \\ 0 & -(1/12)e^{-(t-s)/12} \end{pmatrix}.$$

Thus we have

$$\begin{aligned} |Ae^{A(t-s)}| &= \sup\{|Ez|_0 : |z|_0 = 1\} \\ &= \sup\{(|x|/4)e^{-(t-s)/4} + (|y|/12)e^{-(t-s)/12} : |x| + |y| = 1\} \\ &\leq (1/4)e^{-(t-s)/4} + (1/12)e^{-(t-s)/12}. \end{aligned}$$

Finally we obtain

$$\int_0^t |Ae^{A(t-s)}|ds \leq \int_0^t ((1/4)e^{(s-t)/4} + (1/12)e^{(s-t)/12})ds$$

$$= 2 - e^{-t/4} - e^{-t/12} < 2, t \geq 0,$$

which together with $br = 1/3$, implies that Assumption (4.8) holds. Thus, by Theorem 4.1, every solution of Equation (4.9) tends to 0 as $t \to \infty$.

REMARK. In Example 4.2.7 in Burton [1; p. 253], $br < a$ is necessary for asymptotic stability of the zero solution of Equation (4.1). But in Example 4.1, $a = br = 1/3$, and hence, the result in [1] is not applicable to Equation (4.1) with $a = 1/3$, $b = 1/48$, and $r = 16$.

PART II

Boundedness, Periodic Solutions, and Asymptotic Stability: Two More Fixed Point Theorems

5. INTRODUCTION

Let $a : [0, \infty) \to (0, \infty)$ and $b : [0, \infty) \to R$ be continuous, $r_1 \geq 0$ be constant, $r_2 : [0, \infty) \to [0, \gamma]$ for $\gamma > 0$ be continuous, and consider the scalar equation

$$x'(t) = -a(t)x(t - r_1) + b(t)x^{1/3}(t - r_2(t)) \tag{5.1}$$

with constants $0 < \beta < 1$ and $K > 0$ satisfying

$$\sup_{t \geq 0} |b(t)/a(t + r_1)| \leq \beta \tag{5.2}$$

and if $|t_2 - t_1| \leq 1$, then

$$|\int_{t_1}^{t_2} a(u + r_1)du| \leq K|t_2 - t_1|, \tag{5.3}$$

while for $t \geq 0$ we have

$$2 \sup_{t \geq 0} \int_{t - r_1}^{t} a(u + r_1)du + \beta < 1. \tag{5.4}$$

From existence theory we can conclude that for each continuous initial function $\psi : [-r, 0] \to R$ (where $r \geq \max[r_1, \gamma]$) there is a continuous solution $x(t, 0, \psi)$ on an interval $[0, T)$ for some $T > 0$ and $x(t, 0, \psi) = \psi(t)$ on $[-r, 0]$.

When the r_i are positive, then this becomes a formidable problem. But if $r_1 = r_2 = 0$, then (5.1) is an elementary Bernoulli equation which can be explicitly integrated for exact solutions. Several interesting types of behavior can be distinguished in that elementary case and these lead us to conjectures when $r_i > 0$. Cases of interest are:

i) The zero solution of

$$x' = -2x + x^{1/3}$$

is unstable, but all solutions are bounded.

ii) The zero solution of

$$x' = \frac{-2t}{t^2+1}x + \frac{x^{1/3}}{t^2+1}$$

is asymptotically stable; although the unstable term $x^{1/3}$ dominates the stable term $-x$, the coefficient $\frac{t}{t^2+1}$ strongly dominates the coefficient $\frac{1}{t^2+1}$ and that plays an important role in compactness arguments given later.

iii) The forced equation

$$x' = -2x + x^{1/3} + \sin t$$

has a periodic solution.

We will use fixed point theory to obtain counterparts of these properties where $r_i > 0$. In particular, for moderate size r_i, fixed point theory can treat the formidable case almost as easily as integration can treat the elementary case.

6. A FIRST BOUNDEDNESS RESULT

Krasnoselskii [7] (cf. Smart [11; p. 31]) studied a 1932 paper of Schauder [9] and concluded that the inversion of a perturbed differential operator yields the sum of a contraction and compact map. This is, of course, a great over simplification. The inversion must be done in a cerain way and the space for compactness must be carefully chosen. We will be using that result for our first and last theorems here so it is stated for reference as follows.

THEOREM (Krasnoselskii). Let M be a closed convex non-empty subset of a Banach space $(S, \|\cdot\|)$. Suppose that A and B map M into S such that

(i) $Ax + By \in M (\forall x, y \in M)$,

(ii) A is continuous and AM is contained in a compact set,

(iii) B is a contraction with constant $\alpha < 1$.

Then there is a $y \in M$ with $Ay + By = y$.

It may be noted that for $r_i = 0$ Equation (5.1) is a Bernoulli equation and the zero solution may not be unique to the right if $b > 0$; hence, stability is impossible. Moreover, our method of proof does not distinguish $b > 0$ from $b < 0$. The most we can hope for is bounded solutions. Not only do we obtain boundedness, but the proof and construction is very interesting.

THEOREM 6.1. If (5.2)-(5.4) hold and if ψ is a given continuous initial function which is sufficiently small, then there is a solution $x(t, 0, \psi)$ on $[0, \infty)$ with $|x(t, 0, \psi)| < 1$.

Proof. Let $\psi : [-r, 0] \to R$ be a given continuous initial function with $|\psi(t)| \leq \Psi$ where Ψ is a positive constant to be determined, $\Psi < 1$. Let $h : [-r, \infty) \to [1, \infty)$ be any strictly increasing and continuous function with $h(-r) = 1$, $h(s) \to \infty$ as $s \to \infty$, together with a constant $\alpha < 1$, such that

$$2 \int_{t-r_1}^{t} [a(s + r_1)h(s)/h(t)]ds \leq \alpha. \tag{6.1}$$

Indeed, from (5.4) we have for any such function h that

$$2 \int_{t-r_1}^{t} [a(s + r_1)h(s)/h(t)]ds \leq 2 \int_{t-r_1}^{t} [a(s + r_1)h(t)/h(t)]ds < 1.$$

Let $(S, |\cdot|_h)$ be the Banach space of continuous $\phi : [-r, \infty) \to R$ with

$$|\phi|_h := \sup_{t \geq -r} |\phi(t)/h(t)| < \infty$$

and let $(M, |\cdot|_h)$ be the complete metric space of $\phi \in S$ such that

$$|\phi(t)| \leq 1 \text{ for } -r \leq t < \infty \text{ and } \phi(t) = \psi(t) \text{ on } [-r, 0].$$

Write (5.1) as

$$x'(t) = -a(t + r_1)x(t) + (d/dt) \int_{t-r_1}^{t} a(s + r_1)x(s)ds + b(t)x^{1/3}(t - r_2(t))$$

so that by the variation of parameters formula and integration by parts we have

$$\begin{aligned} x(t) =& x_0 e^{-\int_0^t a(s+r_1)ds} \\ &+ \int_0^t e^{-\int_s^t a(u+r_1)du} [\frac{d}{ds} \int_{s-r_1}^{s} a(u + r_1)x(u)du + b(s)x^{1/3}(s - r_2(s))]ds \end{aligned}$$

or

$$\begin{aligned} x(t) =& x_0 e^{-\int_0^t a(s+r_1)ds} + \int_{t-r_1}^{t} a(u + r_1)x(u)du \\ &- e^{-\int_0^t a(u+r_1)du} \int_{-r_1}^{0} a(u + r_1)x(u)du \\ &- \int_0^t a(s + r_1)e^{-\int_s^t a(u+r_1)du} \int_{s-r_1}^{s} a(u + r_1)x(u)duds \\ &+ \int_0^t b(s)e^{-\int_s^t a(u+r_1)du} x^{1/3}(s - r_2(s))ds \end{aligned} \tag{6.2}$$

where $x(t) = \psi(t)$ on $[-r, 0]$ and $\psi(0) = x_0$.

Define mappings $A, B : M \to M$ by $\phi \in M$ implies that

$$(A\phi)(t) = \int_0^t b(s)e^{-\int_s^t a(u+r_1)du}\phi^{1/3}(s - r_2(s))ds \tag{6.3}$$

and

$$\begin{aligned}(B\phi)(t) =& x_0 e^{-\int_0^t a(s+r_1)ds} + \int_{t-r_1}^t a(u+r_1)\phi(u)du \\ & - e^{-\int_0^t a(u+r_1)du}\int_{-r_1}^0 a(u+r_1)\psi(u)du \\ & - \int_0^t a(s+r_1)e^{-\int_s^t a(u+r_1)du}\int_{s-r_1}^s a(u+r_1)\phi(u)duds.\end{aligned} \tag{6.4}$$

We now show that $\phi, \eta \in M$ implies that $A\phi + B\eta \in M$. Let $\|\cdot\|$ be the supremum norm on $[-r, \infty)$ of $\phi \in S$ if ϕ is bounded. Note that if $\phi, \eta \in M$ then

$$\begin{aligned}&|(A\phi)(t) + (B\eta)(t)| \\ &\le \|\psi\|e^{-\int_0^t a(s+r_1)ds} + \|\eta\|\int_{t-r_1}^t a(u+r_1)du \\ &+ e^{-\int_0^t a(u+r_1)du}\|\psi\|\int_{-r_1}^0 a(u+r_1)du + \|\eta\|\sup_{t\ge 0}\int_{t-r_1}^t a(u+r_1)du + \|\phi\|^{1/3}\beta \\ &\le \|\psi\|e^{-\int_0^t a(s+r_1)ds}(1 + \int_{-r_1}^0 a(u+r_1)du) + 2\sup_{t\ge 0}\int_{t-r_1}^t a(u+r_1)du + \beta \\ &< 1\end{aligned}$$

provided that $\|\psi\|$ is sufficiently small.

Next, we show that AM is equicontinuous. If $\phi \in M$ and if $0 \le t_1 < t_2 < t_1 + 1$

then

$$
\begin{aligned}
&|(A\phi)(t_2) - (A\phi)(t_1)| \\
&\le \Big| \int_0^{t_2} b(s) e^{-\int_s^{t_2} a(u+r_1)du} \phi^{1/3}(s - r_2(s)) ds \\
&- \int_0^{t_1} b(s) e^{-\int_s^{t_1} a(u+r_1)du} \phi^{1/3}(s - r_2(s)) ds \Big| \\
&= \Big| \int_{t_1}^{t_2} b(s) e^{-\int_s^{t_2} a(u+r_1)du} \phi^{1/3}(s - r_2(s)) ds \\
&+ \int_0^{t_1} b(s) \Big[e^{-\int_s^{t_2} a(u+r_1)du} - e^{-\int_s^{t_1} a(u+r_1)du} \Big] \phi^{1/3}(s - r_2) ds \Big| \\
&\le \beta \int_{t_1}^{t_2} a(s + r_1) ds \\
&+ \int_0^{t_1} |b(s)| \Big| e^{-\int_0^{t_2} a(u+r_1)du} - e^{-\int_0^{t_1} a(u+r_1)du} \Big| e^{\int_0^s a(u+r_1)du} ds \\
&\le \beta K|t_2 - t_1| + \beta |e^{-\int_0^{t_2} a(u+r_1)du} - e^{-\int_0^{t_1} a(u+r_1)du}| [e^{\int_0^{t_1} a(u+r_1)du} - 1] \\
&\le \beta K|t_2 - t_1| + \beta |e^{-\int_{t_1}^{t_2} a(u+r_1)du} - 1| \\
&\le \beta K|t_2 - t_1| + \beta \Big| \int_{t_1}^{t_2} a(u + r_1) du \Big| \\
&\le 2\beta K|t_2 - t_1|,
\end{aligned}
$$

by (5.3). In the above calculation, the error made by replacing $e^{-\int_{t_1}^{t_2} a(u+r_1)du}$ by 1 is less than the next term in its series, $|\int_{t_1}^{t_2} a(u + r_1)du|$. In the space $(S, |\cdot|_h)$, the set AM resides in a compact set. To see this, one may consult Burton [1; p. 169], Example 3.1.6(c).

Now we show that there is an $\alpha < 1$ so that $B : M \to M$ and

$$|B\phi_1 - B\phi_2|_h \le \alpha |\phi_1 - \phi_2|_h.$$

We have

$$
\begin{aligned}
&|(B\phi_1)(t) - (B\phi_2)(t)|/h(t) \\
&\le \int_{t-r_1}^t a(u + r_1)|\phi_1(u) - \phi_2(u)|/h(t) du \\
&+ (1/h(t)) \int_0^t a(s + r_1) e^{-\int_s^t a(u+r_1)du} \int_{s-r_1}^s a(u + r_1)|\phi_1(u) - \phi_2(u)| du ds \\
&\le \sup_{t \ge 0} \int_{t-r_1}^t [a(s + r_1)h(s)/h(t)] ds |\phi_1 - \phi_2|_h \\
&+ |\phi_1 - \phi_2|_h \sup_{t \ge 0} \int_{t-r_1}^t a(u + r_1)[h(u)/h(t)] du
\end{aligned}
$$

$$\leq 2|\phi_1 - \phi_2|_h \sup_{t\geq 0} \int_{t-r_1}^{t} a(u+r_1)[h(u)/h(t)]du$$
$$\leq \alpha|\phi_1 - \phi_2|_h$$

for some $\alpha < 1$ by (5.4).

Finally, we need to show A continuous.

Let $\epsilon > 0$ be given and let $\phi \in M$. We must find $\delta > 0$ such that $[\eta \in M, |\phi - \eta|_h < \delta]$ implies that $|A\phi - A\eta|_h < \epsilon\beta$.

Now $x^{1/3}$ is uniformly continuous on $[-1, 1]$ so for $\epsilon > 0$ and a fixed $T > 0$ with $4/h(T) < \epsilon$ there is a $\delta > 0$ such that $|x_1 - x_2| < \delta h(T)$ implies $|x_1^{1/3} - x_2^{1/3}| < \epsilon/2$. Thus for $|\phi(t) - \eta(t)| < \delta h(t)$ and for $t > T$ we have

$$\begin{aligned}
&|(A\phi)(t) - (A\eta)(t)|/h(t) \\
&\leq (1/h(t)) \int_0^t b(s)e^{-\int_s^t a(u+r_1)du}|\phi^{1/3}(s - r_2) - \eta^{1/3}(s - r_2(s))|ds \\
&\leq (1/h(t)) \bigg[\int_0^T |b(s)|e^{-\int_s^t a(u+r_1)du}|\phi^{1/3}(s - r_2) - \eta^{1/3}(s - r_2(s))|ds \\
&+ \int_T^t 2|b(s)|e^{-\int_s^t a(u+r_1)du}ds\bigg] \\
&\leq [(\beta\epsilon)/(2h(t))] \int_0^T a(s+r_1)e^{-\int_s^t a(u+r_1)du}ds + (2\beta/h(T)) \\
&\leq (\beta\epsilon/2) + (2\beta/h(T)) \\
&< \epsilon\beta.
\end{aligned}$$

The conditions of Krasnoselskii's theorem are satisfied and there is a fixed point.

7. Boundedness and Periodicity

Stability, boundedness, and the existence of periodic solutions are concepts which are closely related. This is discussed in detail in Chapter 0 of Burton [1] in the context of Liapunov's direct method. Given a system $x' = f(t, x)$ and a Liapunov function $V(t, x)$ which is radially unbounded and decrescent, whose derivative along solutions is negative definite, and which satisfies a global Lipschitz condition in x, that same Liapunov function will prove uniform ultimate boundedness for solutions of a perturbed equation, $x' = f(t, x) + p(t)$ when p is bounded and small enough relative to the Lipschitz condition and the derivative of V along solutions of the original system.

Here we study the same thing in the context of fixed point theory and essentially the same kind of properties hold. All of the work in Section 6 can be used to prove that a perturbed form of (5.1) has a periodic solution using Schaefer's fixed point theorem. For reference we have (Smart [11; p. 29])

THEOREM (Schaefer). Let $(\mathcal{B}, \|\cdot\|)$ be a normed space, H a continuous mapping of $\mathcal{B}$ into $\mathcal{B}$ which is compact on each bounded subset X of $\mathcal{B}$. Then either

(i) the equation $x = \lambda Hx$ has a solution for $\lambda = 1$, or

(ii) the set of all such solutions x, for $0 < \lambda < 1$, is unbounded.

Consider Equation (5.1) again with a perturbation

$$x'(t) = -a(t)x(t-r_1) + b(t)x^{1/3}(t - r_2(t)) + p(t) \tag{7.1}$$

in which $p : R \to R$ is continuous and there is a $T > 0$ with

$$r_2(t+T) = r_2(t), p(t+T) = p(t), a(t+T) = a(t), \text{ and } b(t+T) = b(t) \tag{7.2}$$

for all $t \in R$. We will need relaxed forms of (5.2)-(5.4). Assume there are positive constants β and K with

$$\sup_{t\geq 0} |b(t)/a(t+r_1)| \leq \beta \tag{7.3}$$

and if $|t_2 - t_1| \leq 1$, then

$$|\int_{t_1}^{t_2} a(u+r_1)du| \leq K|t_2 - t_1|, \tag{7.4}$$

while for $t \geq 0$ we have

$$2\sup_{t\geq 0} \int_{t-r_1}^{t} a(u+r_1)du < 1. \tag{7.5}$$

THEOREM 7.1. If (7.2) - (7.5) hold, then (7.1) has a T-periodic solution.

Proof. Write (7.1) as

$$x' = -a(t+r_1)x(t) + \frac{d}{dt}\int_{t-r_1}^{t} a(s+r_1)x(s)ds + b(t)x^{1/3}(t - r_2(t)) + p(t)$$

or

$$(xe^{\int_0^t a(s+r_1)ds})' = e^{\int_0^t a(s+r_1)ds}\left[\frac{d}{dt}\int_{t-r_1}^{t} a(s+r_1)x(s)ds + b(t)x^{1/3}(t-r_2(t)) + p(t)\right].$$

Note that since the function a is periodic and $a(t) > 0$ we have $\int_0^t a(s)ds \uparrow \infty$ as $t \to \infty$. If we consider only solutions of (7.1) bounded on R (if any), we can integrate from $-\infty$ to t and obtain (after integration by parts):

$$
\begin{aligned}
&x(t)\\
&= \int_{t-r_1}^{t} a(u+r_1)x(u)du - \int_{-\infty}^{t} e^{-\int_s^t a(u+r_1)du} a(s+r_1) \int_{s-r_1}^{s} a(u+r_1)x(u)du\\
&+ \int_{-\infty}^{t} e^{-\int_s^t a(u+r_1)du}[b(s)x^{1/3}(s-r_2(s)) + p(s)]ds.
\end{aligned}
$$

Now, if $(P_T, \|\cdot\|)$ is the Banach space of continuous T-periodic functions with the supremum norm, then we can define an operator $H : P_T \to P_T$ by $\phi \in P_T$ implies that

(7.6)

$$
\begin{aligned}
&(H\phi)(t)\\
&= \lambda\Big[\int_{t-r_1}^{t} a(u+r_1)\phi(u)du - \int_{-\infty}^{t} e^{-\int_s^t a(u+r_1)du} a(s+r_1) \int_{s-r_1}^{s} a(u+r_1)\phi(u)du\\
&+ \int_{-\infty}^{t} e^{-\int_s^t a(u+r_1)du}[b(s)\phi^{1/3}(s-r_2) + p(s)]ds\Big]
\end{aligned}
$$

where $0 < \lambda \leq 1$.

We now obtain an *a priori* bound on all fixed points of H. If $x \in P_T$ is a fixed point, then using $\|\cdot\|$ to denote the supremum norm we have $Q > 0$ with

$$\|x\| \leq \sup_{t\in R} \int_{t-r_1}^{t} a(u+r_1)du[2\|x\|] + \|x\|^{1/3} \sup_{t\in R} \frac{|b(t)|}{a(t+r_1)} + Q.$$

Taking into account (7.3)-(7.5), this will establish an *a priori* bound if

$$2\sup \int_{t-r_1}^{t} a(u+r_1)du \leq \alpha < 1.$$

An argument just like that in the proof of Theorem 6.1 will show that H maps bounded sets into equicontinuous sets. Thus, H maps bounded sets into compact subsets of our space. By Schaefer's theorem there is a fixed point and it solves (7.1). This completes the proof.

8. ASYMPTOTIC STABILITY

This section is motivated by the Bernoulli equation

$$x'(t) = -(2/t)x(t) - (1/t^2)x^3(t).$$

Even without solving the equation, the investigator would conjecture that the asymptotically stable part

$$x' = -(2/t)x$$

would dominate the term $(1/t^2)x^3$ since x dominates x^3 for small x and $1/t^2$ is $L^1[1,\infty)$.

But that entire rationale vanishes when we consider

$$x'(t) = -a(t)x(t-r_1) + b(t)x^{1/3}(t-r_2(t)) \tag{8.1}$$

again where the exact conditions of Theorem 6.1 hold except (5.2) becomes: There is a $\beta < 1$ with

$$\sup_{t\geq 0} |b(t)/a(t+r_1)| \leq \beta \text{ and } b(t)/a(t+r_1) \to 0 \text{ as } t \to \infty. \tag{5.2*}$$

Not only does $x^{1/3}$ dominate x for small x, but there is a delay in the linear part which makes it a nontrivial stability problem and $b(t)$ is not assumed to be $L^1[0,\infty)$.

Thus, the investigator recognizes that it is a challenge to prove asymptotic stability. Our point here is that fixed point theory is such a powerful tool that this becomes a simple problem.

THEOREM 8.1. Let (5.2*), (5.3), and (5.4) hold and assume that $\int_0^\infty a(s)ds = \infty$. If ψ is a given initial funciton which is sufficiently small, then (8.1) has a solution $x(t,0,\psi) \to 0$ as $t \to \infty$.

Proof. All of the calculations of the proof of Theorem 6.1 hold with $h(t) = 1$ when $|\cdot|_h$ is replaced by the supremum norm $\|\cdot\|$.

Define $g : [0,\infty) \to (0,1]$ by

$$g(t) = \int_0^t a(s+r_1)e^{-\int_s^t a(u+r_1)du}[b(s)/a(s+r_1)]ds. \tag{8.2}$$

We can easily modify the classical proof that the convolution of an L^1-function [substitute $a(s+r_1)e^{-\int_s^t a(u+r_1)}du$] with a function tending to zero $[b(s)/a(s+r_1)]$ tends to zero to show that $g(t) \to 0$ as $t \to \infty$.

Add to M the condition that $\phi \in M$ implies that $\phi(t) \to 0$ as $t \to \infty$. With A defined by (6.3) we see that for $\phi \in M$ then $|(A\phi)(t)| \leq g(t)$. Clearly $(B\phi)(t) \to 0$ as $t \to \infty$.

Since AM has been shown to be equicontinuous, A maps M into a compact subset of M. By Krasnoselskii's theorem there is a $y \in M$ with $Ay + By = y$. As $y \in M$, $y(t) \to 0$ as $t \to \infty$. This completes the proof.

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