A LIAPUNOV FUNCTIONAL FOR A SINGULAR INTEGRAL EQUATION

T. A. BURTON

Abstract. We consider a scalar integral equation $x(t) = a(t) - \int_0^t C(t,s)g(s,x(s))ds$ where $a \in L^2[0,\infty)$, while $C(t,s)$ has a significant singularity, but is convex when $t - s > 0$. We construct a Liapunov functional and show that $g(t,x(t)) - a(t) \in L^2[0,\infty)$ and that $x(t) - a(t) \to 0$ pointwise as $t \to \infty$. Small perturbations are also added to the kernel. In addition, we consider both infinite and finite delay problems.

1. Introduction

The first results on Liapunov functionals for integral equations (which were not converted to differential equations) were obtained in 1992 [1] and most of the subsequent work was recently collected in [5]. To this point it can be mainly regarded as "in between" theory. In the classical theory [11; Chapter VI] if an equation

$$x(t) = a(t) - \int_0^t C(t,s)g(s,x(s))ds \tag{1}$$

can be differentiated then Liapunov's direct method can be readily applied. The work in [5] goes one step further by applying the method to (1) directly without differentiation; however, in all of that work it is required that the kernel be continuous. Hence, it lies between the classical method and the work with discontinuous kernels such as is found in a heat equation of the form

$$x(t) = a(t) - \int_0^t \frac{1}{\sqrt{\pi(t-s)}} g(s,x(s))ds.$$

(See Miller [11; p. 209, second equation].)

This paper offers a first step toward treating discontinuous kernels by means of Liapunov functionals. We will focus on kernels which satisfy a truncated convexity condition of a type very common in problems where partial differential equations are converted to integral equations.

There are five main conditions which we require and we will discuss them under the heading of **critique** later. First, for each continuous

1991 *Mathematics Subject Classification.* Primary: 45D05, 45M10.

Key words and phrases. Integral equations, Liapunov functionals.

function x, then $\int_0^t C(t,s)g(s,x(s))ds$ exists. Next, for each small $\epsilon > 0$ then

$$C(t,s) \geq 0, C_s(t,s) \geq 0, C_{st}(t,s) \leq 0, C_t(t,s) \leq 0 \tag{2}$$

provided that

$$0 \leq s \leq t - \epsilon,\ t < \infty; \tag{3}$$

thus, $t - s \geq \epsilon$. The kernel $(t-s)^{-p}$ for $0 < p < 1$ does satisfy that condition. The final two conditions are (8) and (9) to be given later.

Convex kernels (without (3)) were discussed by Volterra in 1928 [12] and have been widely used since then in mathematical biology, viscoelasticity, circulating fuel nuclear reactors, neural networks, and many other places. (See [5] for discussion and references.)

Volterra suggested that a Liapunov functional could be constructed for

$$x' = -\int_0^t C(t,s)g(x(s))ds$$

when C is convex and $xg(x) > 0$ if $x \neq 0$. Levin [10] accomplished that construction in 1963 (continuing the work for many years) and we did it for (1) in 1992 [1], using the more general $g(t,x)$. It is the latter functional which we modify here for the singular kernel.

2. Singular perturbations

One of the simple and common examples of (2) (without (3)) is

$$C(t,s) = [\delta^2 + t - s]^{-1/2},\ \delta^2 > 0.$$

In a series of papers [1-4, 6-8] we have studied (1) with such kernels (not necessarily of convolution type) and find that if $a \in L^2[0,\infty)$, then $x \in L^2[0,\infty)$, $x - a \in L^2[0,\infty)$, and that $x(t) \to a(t)$ pointwise as $t \to \infty$. There are further parallel results concerning the relation of the resolvent to the kernel and a substitute variation of parameters formula using the kernel in place of the resolvent. The proofs are uniformly the same for every $\delta^2 > 0$. If we perturb C into a singular kernel, will those L^2 and pointwise properties still hold and will the proofs be essentially the same? With one little trick, everything stays the same.

This is simply the next step in the study of convex kernels which are so often used in real-world problems. In an earlier series of papers we questioned the possibility of measuring any real-world phenomena so closely as to verify (2). Thus, we perturbed $C(t,s)$ to $C(t,s) + D(t,s)$ where C satisfied (2) (without (3)) and

$$\int_0^\infty |D(u+t,t)|du \leq \delta,\ \int_0^t |D(t,s)|ds \leq \gamma,\ \delta + \gamma < 2.$$

We modified the Liapunov functional and showed that the L^2 and pointwise behavior still obtained. The study lent integrity to the practice of modeling real-world problems with kernels satisfying (2) even when the intricacies of (2) could not be measured.

Our study here takes the next step and shows that perturbing $C(t,s)$ into a singular kernel still preserves the qualitative properties of L^2 and pointwise convergence of x and a.

3. A Liapunov functional

Consider the scalar equation (1) with (2) and (3) satisfied and let both $a(t)$ and $g(t,x)$ be continuous for $t \geq 0$ and $x \in \Re$, while

$$xg(t,x) > 0 \text{ if } x \neq 0. \tag{4}$$

For an $\epsilon > 0$ and for $t \geq \epsilon$ we define a Liapunov functional by

$$V(t,\epsilon) = \int_0^{t-\epsilon} C_s(t,s)\bigg(\int_s^t g(u,x(u))du\bigg)^2 ds + C(t,0)\bigg(\int_0^t g(u,x(u))du\bigg)^2. \tag{5}$$

While $C_s(t,s)$ may be badly behaved at $s = t$, we have $s \leq t - \epsilon$ or $\epsilon \leq t - s$ so the bad point is always avoided in $V(t,\epsilon)$. We will discuss existence later in this section.

Theorem 3.1. *Let x be a continuous solution of (1) on $[0,\infty)$ and let (2) and (3) be satisfied. If $\epsilon > 0$ is chosen and if $V(t,\epsilon)$ is defined in (5) then for $t \geq \epsilon$ we have*

$$\begin{aligned} \frac{dV(t,\epsilon)}{dt} &\leq 2g(t,x(t))\bigg[a(t) - x(t) + C(t,t-\epsilon)\int_{t-\epsilon}^t g(u,x(u))du \\ &\quad - \int_{t-\epsilon}^t C(t,s)g(s,x(s))ds\bigg] + C_s(t,t-\epsilon)\bigg(\int_{t-\epsilon}^t g(u,x(u))du\bigg)^2. \end{aligned} \tag{6}$$

Proof. For $t \geq \epsilon$ we have $C_t(t,0) \leq 0$ and $C_{st}(t,s) \leq 0$ when $0 \leq s \leq t - \epsilon$ so by Leibnitz's rule we have

$$\begin{aligned} V'(t,\epsilon) &\leq C_s(t,t-\epsilon)\bigg(\int_{t-\epsilon}^t g(u,x(u))du\bigg)^2 \\ &\quad + 2g(t,x(t))\int_0^{t-\epsilon} C_s(t,s)\int_s^t g(u,x(u))duds \\ &\quad + 2g(t,x(t))C(t,0)\int_0^t g(u,x(u))du. \end{aligned}$$

Integrating the next-to-last term by parts yields

$$\begin{aligned}
&2g(t,x(t))\int_0^{t-\epsilon} C_s(t,s)\int_s^t g(u,x(u))duds\\
&= 2g(t,x(t))\left[C(t,s)\int_s^t g(u,x(u))du\Big|_0^{t-\epsilon} + \int_0^{t-\epsilon} C(t,s)g(s,x(s))ds\right]\\
&= 2g(t,x(t))\left[C(t,t-\epsilon)\int_{t-\epsilon}^t g(u,x(u))du - C(t,0)\int_0^t g(u,x(u))du\right.\\
&\left.+\int_0^{t-\epsilon} C(t,s)g(s,x(s))ds\right].
\end{aligned}$$

Thus,

$$\begin{aligned}
V'(t,\epsilon) &\le C_s(t,t-\epsilon)\left(\int_{t-\epsilon}^t g(u,x(u))du\right)^2\\
&+ 2g(t,x(t))\left[C(t,t-\epsilon)\int_{t-\epsilon}^t g(u,x(u))du + \int_0^{t-\epsilon} C(t,s)g(s,x(s))ds\right]\\
&= C_s(t,t-\epsilon)\left(\int_{t-\epsilon}^t g(u,x(u))du\right)^2\\
&+ 2g(t,x(t))\left[C(t,t-\epsilon)\int_{t-\epsilon}^t g(u,x(u))du - \int_{t-\epsilon}^t C(t,s)g(s,x(s))ds\right]\\
&+ 2g(t,x(t))[a(t)-x(t)],
\end{aligned}$$

as required. We have used the integral equation (1) in the last step. □

Three relations will be needed for us to parlay this Liapunov functional derivative into a qualitative result for a solution of (1). First, we must be able to estimate the relation between $xg(t,x)$ and $g^2(t,x)$. Our conditions (8) and (9) below will allow such strongly singular kernels that it is not a great surprise to need $g(t,x)$ bounded by a linear function. We ask that

$$xg(t,x) \ge g^2(t,x). \tag{7}$$

Next, we need some control over the magnitude of the singularity and this turns out to be enlightening. In the simplest case it asks for a locally L^1 kernel.

We suppose that there are positive constants α and β with $\alpha+\beta<1$ so that there is an $\epsilon>0$ with

$$\int_s^{s+\epsilon}[\epsilon C_s(u,u-\epsilon)+C(u,u-\epsilon)+|C(u,s)|]du < \alpha \tag{8}$$

for $0\le s<\infty$ and that

$$C(t,t-\epsilon)\epsilon + \int_{t-\epsilon}^t |C(t,s)|ds < \beta \tag{9}$$

for $\epsilon \le t < \infty$. Note in (2) and (3) that we do not specify the sign of $C(s,s)$ so the absolute value is needed in (8).

Remark We will use the $\alpha+\beta<1$ relation in obtaining the second line of (10). In fact, we can get by with $\alpha+\beta<2$, as we actually do using ξ and M in the proof of Theorem 5.2. This, however, will complicate the relation $a-g \in L^2$ which we want to preserve. Theorem 5.2 will, thereby, contain this possible improvement of Theorem 3.2 by taking $D=0$.

Critique

Conditions (8) and (9) allow for gross singularities to occur at $s=t$. We have already noticed that $C_s(u,u-\epsilon)$ occurring in the derivative of V is always well away from the singularity and (2) holds for it. While (8) and (9) occur as technical necessities in a later computation, repeated again and again in this work here, one would really like at least a theoretical rational for them. To begin with, for mild singularities such as $C(t,s)=[t-s]^{-p}$ for $0<p<1$, those conditions are too lenient to make any sense; for in that case $\alpha+\beta \to 0$ as $\epsilon \to 0$. Remarks and references in Kirk and Olmstead [9] suggest impulse functions, in which case $\alpha+\beta$ would not tend to zero.

Concerning existence theory, mild singularities such as $C(t,s) = [t-s]^{-p}$ for $0<p<1$ offer no existence problems at all when g is Lipschitz since in contraction mapping arguments $\int_0^t C(t,s)g(s,\phi(s))ds$ is continuous whenever ϕ is continuous. We use a weighted norm to get existence on an arbitrary interval $[0,T]$. See Windsor [13] for a simple and recent treatment of existence in the presence of such singularities. Miller [11] gives very general conditions for existence of solutions and there are other special existence results scattered throughout the literature. It would be a distraction and a limitation to repeat them here. To keep the focus on what is fundamentally new here, we simply work with problems in which a solution is known to exist.

Theorem 3.2. *Let x be a continuous solution of (1) on $[0,\infty)$ and let (2), (3), (7)–(9) hold. If, in addition, $a \in L^2[0,\infty)$ so are $g(t,x(t))$ and $g(t,x(t))-a(t)$.*

Proof. We begin by organizing the derivative of V which we computed in (6). First, by the Schwarz inequality we have

$$C_s(t,t-\epsilon)\left(\int_{t-\epsilon}^t g(u,x(u))du\right)^2 \le \epsilon C_s(t,t-\epsilon)\int_{t-\epsilon}^t g^2(u,x(u))du.$$

Next,

$$|2g(t,x)C(t,t-\epsilon)\int_{t-\epsilon}^t g(u,x(u))du| \le C(t,t-\epsilon)\int_{t-\epsilon}^t [g^2(t,x(t))+g^2(u,x(u))]du$$

and

$$|2g(t,x)\int_{t-\epsilon}^{t} C(t,s)g(s,x(s))ds| \le \int_{t-\epsilon}^{t} C(t,s)[g^2(t,x(t))+g^2(s,x(s))]ds.$$

These three relations in (6) yield

$$\begin{aligned} V'(t,\epsilon) &\le 2g(t,x(t))[a(t)-x(t)] \\ &\quad + C(t,t-\epsilon)\epsilon g^2(t,x(t)) + g^2(t,x(t))\int_{t-\epsilon}^{t} C(t,s)ds \\ &\quad + \int_{t-\epsilon}^{t} [\epsilon C_s(t,t-\epsilon) + C(t,t-\epsilon) + C(t,s)]g^2(s,x(s))ds. \end{aligned}$$

By (7) we obtain

$$\begin{aligned} 2g(t,x(t)[a(t)-x(t)] &\le 2g(t,x(t))a(t) - 2g^2(t,x(t)) \\ &= -g^2(t,x(t)) - (g(t,x(t)) - a(t))^2 + a^2(t). \end{aligned} \tag{10}$$

Invoke (8) and (9) to find ϵ, α, and β. Using (9) we have

$$\begin{aligned} V'(t,\epsilon) &\le a^2(t) - (g(t,x) - a(t))^2 - (1-\beta)g^2(t,x) \\ &\quad + \int_{t-\epsilon}^{t} [\epsilon C_s(t,t-\epsilon) + C(t,t-\epsilon) + C(t,s)]g^2(s,x(s))ds. \end{aligned} \tag{11}$$

Take $|C(t,s)|$ in this expression, integrate from ϵ to t, and work with the last term, interchanging the order of integration. We have

$$\begin{aligned} &\int_{\epsilon}^{t}\int_{u-\epsilon}^{u} [\epsilon C_s(u,u-\epsilon) + C(u,u-\epsilon) + |C(u,s)|]g^2(s,x(s))dsdu \\ &\le \int_{0}^{t}\int_{s}^{s+\epsilon} [\epsilon C_s(u,u-\epsilon) + C(u,u-\epsilon) + |C(u,s)|]du g^2(s,x(s))ds \\ &\le \alpha \int_{0}^{t} g^2(s,x(s))ds \end{aligned}$$

using (8). This now yields

$$\begin{aligned} V(t,\epsilon) \le V(\epsilon,\epsilon) &+ \int_{\epsilon}^{t} a^2(u)du - \int_{\epsilon}^{t} (g(s,x(s)) - a(s))^2 ds \\ &-(1-\alpha-\beta)\int_{\epsilon}^{t} g^2(u,x(u))du + \alpha\int_{0}^{\epsilon} g^2(u,x(u))du. \end{aligned}$$

We assumed $x(t)$ exists so $V(\epsilon,\epsilon)$ is finite, while $V(t,\epsilon) \ge 0$. Put the negative terms on the left to finish the proof. □

We now want to investigate two things. First, can we get boundedness "at the same price?" Next, can we add just a bit to the conditions and actually show that $x(t) \to a(t)$ as $t \to \infty$?

Theorem 3.3. *If x is a continuous solution of (1) on $[0,\infty)$, then for $t \geq \epsilon$*

$$(x(t)-a(t))^2 \leq 6\epsilon C^2(t,t-\epsilon)\int_{t-\epsilon}^{t} g^2(u,x(u))du$$
$$+6C(t,t-\epsilon)V(t,\epsilon)+2\int_{t-\epsilon}^{t} C^2(t,s)ds\int_{t-\epsilon}^{t} g^2(s,x(s))ds.$$

Proof. In the calculation below, it will save much work to set

$$H := 2\Big(\int_{t-\epsilon}^{t} C(t,s)g(s,x(s))ds\Big)^2.$$

Starting with (1) we have

$$(x(t)-a(t))^2 = \Big(\int_0^t C(t,s)g(s,x(s))ds\Big)^2$$
$$\leq 2\Big(\int_0^{t-\epsilon} C(t,s)g(s,x(s))ds\Big)^2 + H$$
$$= 2\Big(-C(t,s)\int_s^t g(u,x(u))du\Big|_0^{t-\epsilon}$$
$$+\int_0^{t-\epsilon} C_s(t,s)\int_s^t g(u,x(u))duds\Big)^2 + H$$
$$= 2\Big(-C(t,t-\epsilon)\int_{t-\epsilon}^t g(u,x(u))du + C(t,0)\int_0^t g(u,x(u))du$$
$$+\int_0^{t-\epsilon} C_s(t,s)\int_s^t g(u,x(u))duds\Big)^2 + H$$
$$\leq 6\Big(-C(t,t-\epsilon)\int_{t-\epsilon}^t g(u,x(u))du\Big)^2 + 6\Big(C(t,0)\int_0^t g(u,x(u))du\Big)^2$$
$$+6\int_0^{t-\epsilon} C_s(t,s)ds\int_0^{t-\epsilon} C_s(t,s)\Big(\int_s^t g(u,x(u))du\Big)^2 ds + H$$
$$\leq 6\Big(-C(t,t-\epsilon)\int_{t-\epsilon}^t g(u,x(u))du\Big)^2$$
$$+6\big[C(t,0)+\int_0^{t-\epsilon} C_s(t,s)ds\big]V(t,\epsilon)+H.$$
$$= 6C^2(t,t-\epsilon)\epsilon\int_{t-\epsilon}^t g^2(u,x(u))du + 6C(t,t-\epsilon)V(t,\epsilon)+H.$$

□

From this we can obtain $x(t)$ bounded in several simple ways.

Notice that in our last condition below we would be asking for a convolution kernel to be in $L^2[0,\infty)$ which is unlike anything else we have asked in this note. In [6] we had offered a better argument in the linear case, but it required a bounded kernel.

Theorem 3.4. *Suppose that x is a continuous solution of (1) on $[0,\infty)$, that $g(t,x(t)) \in L^2[0,\infty)$, that for each large T then*

$$\int_0^T |C(t,s)|ds \to 0$$

as $t \to \infty$, and that

$$\sup_{0\le t} \int_0^t C^2(t,s)ds \le M$$

for some $M > 0$. Then $x(t) \to a(t)$ as $t \to \infty$.

Proof. We have

$$\begin{aligned} |x(t)-a(t)| &\le \int_0^T |C(t,s)||g(s,x(s))|ds \\ &\quad + \sqrt{\int_T^t C^2(t,s)ds \int_T^t g^2(s,x(s))ds} \\ &\le \|g\|^{[0,T]} \int_0^T |C(t,s)|ds + \sqrt{M\int_T^t g^2(s,x(s))ds}. \end{aligned}$$

For a given $\epsilon > 0$, take T so large that the last term is less than $\epsilon/2$. Then we have $J > 0$ with $\|g\|^{[0,T]} \le J$, where the notation means the supremum on $[0,T]$. Finally, if t is large enough then $\int_0^T |C(t,s)|ds \le \epsilon/2J$. □

4. SMALL KERNELS

We are now going to develop Liapunov functionals for integral equations with small kernels. Some of this will later be used to handle a perturbation of the convex kernel C. We begin with the scalar equation

$$x(t) = a(t) - \int_0^t D(t,s)g(s,x(s))ds \tag{12}$$

where g again satisfies (4) and D satisfies several integrability conditions to follow. Again we will be treating an assumed solution, rather than give detailed conditions under which the solution exists.

Theorem 4.1. *Let (4) hold, $|g(t,x)| \le |x|$, D be continuous, $\int_0^\infty |D(u+t,t)|du \le \delta$, $\int_0^t |D(t,s)|ds \le \gamma$, and for p an even positive integer let $a \in L^p[0,\infty)$. If*

$$\delta + (p-1)\gamma - p < 0$$

and if $x(t)$ is a continuous solution of (12) on $[0,\infty)$ then

$$\int_0^\infty g^p(s,x(s))ds < \infty.$$

Proof. Define

$$V(t) = \int_0^t \int_{t-s}^{\infty} |D(u+s,s)| du g^p(s,x(s))ds$$

and compute the derivative. We quickly arrive at

$$\begin{aligned} V'(t) \le \delta g^p(t,x) - \int_0^t |D(t,s)| g^p(s,x(s))ds \\ + pg^{p-1}(t,x(t))[a(t) - x(t) - \int_0^t D(t,s)g(s,x(s))ds]. \end{aligned}$$

That last term is identically zero since x solves (12); this will be clarified in the remark following the proof of Theorem 5.1. Multiplying out that last term and using the inequality (under appropriate conditions) $ab \le (a^p/p) + (b^s/s)$ we can find positive numbers q and M with q as small as we please, while M is large, and show that the last term is bounded by

$$\begin{aligned} & p\left[a(t)g^{p-1}(t,x(t)) - g^{p-1}(t,x(t))x(t) + \int_0^t |D(t,s)|\big[g^{p-1}(t,x(t))g(s,x(s))\big]ds\right] \\ & \le q(p-1)(g^{p-1}(t,x(t)))^{\frac{p}{p-1}} + Ma^p(t) - pg^p(t,x(t)) \\ & + \int_0^t |D(t,s)|\left[(p-1)(g^{p-1}(t,x(t)))^{\frac{p}{p-1}} + g^p(s,x(s))\right]ds \\ & \le q(p-1)g^p(t,x(t)) + Ma^p(t) - pg^p(t,x(t)) + (p-1)\int_0^t |D(t,s)|ds g^p(t,x(t)) \\ & + \int_0^t |D(t,s)|g^p(s,x(s))ds. \end{aligned}$$

We now have

$$V'(t) \le \left[\delta + q(p-1) - p + (p-1)\gamma\right] g^p(t,x(t)) + Ma^p(t).$$

We can choose q so small that we can find $\mu > 0$ and have

$$V'(t) \le -\mu g^p(t,x(t)) + Ma^p(t).$$

An integration yields the result. □

This is a new result, but there are other Liapunov functionals for this equation as seen in [5; pp. 60-64]. They require $\gamma = 1$ for all $p > 1$. This result is much simpler and has more flexibility.

5. Fully perturbed convex kernel

We now consider the scalar equation

$$x(t) = a(t) - \int_0^t [C(t,s) + D(t,s)]g(s,x(s))ds \tag{13}$$

where (2), (3), and (4) hold. We also suppose that there are positive constants γ and δ with

$$\int_0^t |D(t,s)|ds \le \gamma, \ \int_0^\infty |D(u+t,t)|du \le \delta, \ \gamma+\delta<2. \tag{14}$$

Unlike differential equations, we can add kernels and add Liapunov functionals in a completely seamless manner.

Theorem 5.1. *Let (2), (3), and (14) hold for (13) and let D be continuous. Then for*

$$V(t,\epsilon) = \int_0^t \int_{t-s}^\infty |D(u+s,s)|du g^2(s,x(s))ds$$
(15)
$$+\int_0^{t-\epsilon} C_s(t,s)\Big(\int_s^t g(u,x(u))du\Big)^2 ds + C(t,0)\Big(\int_0^t g(u,x(u))du\Big)^2$$

we have

$$\begin{aligned}
V'(t,\epsilon) \le\ & 2g(t,x(t))\big[a(t)-x(t)-\int_0^t D(t,s)g(s,x(s))ds \\
&\quad + C(t,t-\epsilon)\int_{t-\epsilon}^t g(u,x(u))du\big] \\
& - 2g(t,x(t))\int_{t-\epsilon}^t C(t,s)g(s,x(s))ds + C_s(t,t-\epsilon)\Big(\int_{t-\epsilon}^t g(u,x(u))du\Big)^2 \\
& + \delta g^2(t,x(t)) - \int_0^t |D(t,s)|g^2(s,x(s))ds.
\end{aligned}$$

Proof. As $C_t \le 0$ and $C_{st} \le 0$ we have

$$\begin{aligned}
V'(t,\epsilon) \le\ & \int_0^\infty |D(u+t,t)|du g^2(t,x(t)) - \int_0^t |D(t,s)|g^2(s,x(s))ds \\
& + C_s(t,t-\epsilon)\Big(\int_{t-\epsilon}^t g(u,x(u))du\Big)^2 \\
& + 2g(t,x(t))\int_0^{t-\epsilon} C_s(t,s)\int_s^t g(u,x(u))duds \\
& + 2g(t,x(t))C(t,0)\int_0^t g(u,x(u))du
\end{aligned}$$

(Integrate the next-to-last term by parts.)

$$\begin{aligned}
\le\ & \delta g^2(t,x(t)) - \int_0^t |D(t,s)|g^2(s,x(s))ds \\
& + 2g(t,x(t))C(t,0)\int_0^t g(u,x(u))du + C_s(t,t-\epsilon)\Big(\int_{t-\epsilon}^t g(u,x(u))du\Big)^2
\end{aligned}$$

$$+ 2g(t,x(t))\left[C(t,s)\int_s^t g(u,x(u))du\Big|_0^{t-\epsilon} + \int_0^{t-\epsilon} C(t,s)g(s,x(s))ds\right]$$

$$= \delta g^2(t,x(t)) + C_s(t,t-\epsilon)\left(\int_{t-\epsilon}^t g(u,x(u))du\right)^2$$

$$+ 2g(t,x(t))C(t,0)\int_0^t g(u,x(u))du - \int_0^t |D(t,s)|g^2(s,x(s))ds$$

$$+ 2g(t,x(t))\left[C(t,t-\epsilon)\int_{t-\epsilon}^t g(u,x(u))du - C(t,0)\int_0^t g(u,x(u))du\right.$$

$$\left.+ \int_0^{t-\epsilon} C(t,s)g(s,x(s))ds\right]$$

(Write the last term as $\int_0^t C(t,s)g(s,x(s))ds - \int_{t-\epsilon}^t C(t,s)g(s,x(s))ds$ and replace the first of these by its value in (13), thereby linking the Liapunov functional to the integral equation.)

$$= \delta g^2(t,x(t)) + C_s(t,t-\epsilon)\left(\int_{t-\epsilon}^t g(u,x(u))du\right)^2$$

$$- \int_0^t |D(t,s)|g^2(s,x(s))ds + 2g(t,x)C(t,t-\epsilon)\int_{t-\epsilon}^t g(u,x(u))du$$

$$+ 2g(t,x(t))\left[a(t) - x(t) - \int_0^t D(t,s)g(s,x(s))ds\right]$$

$$- 2g(t,x(t))\int_{t-\epsilon}^t C(t,s)g(s,x(s))ds,$$

as required. □

Remark When $D = 0$ then V' coincides with V' in Theorem 3.1. When $C = 0$, then V' coincides with V' in Theorem 4.1 for $p = 2$. The replacement described in the sentence before the last set of expressions in the above proof is well-motivated as it links the Liapunov functional with the integral equation. That work coincides with the rather mysterious addition of the second line in the derivation of V' in the proof of Theorem 4.1.

Caution! In view of this remark we would conjecture that if the conditions of Theorems 3.2 and 4.1 hold for $p = 2$, then a solution of (13) satisfies $g(t,x(t)) \in L^2$ when $a \in L^2$. But this is wrong. Both of those theorems use the x of the equation in the term $2g(t,x)x$. When we add $C(t,s) + D(t,s)$ we only have x once. Thus, instead of asking $\alpha + \beta < 2$, as in the remark following (8) and (9), followed by asking $\delta + \gamma < 2$, as in (14), we are forced to ask $\alpha + \beta + \delta + \gamma < 2$.

Take courage! It is often true. For "mild" singularities, such as $C(t,s) = [t-s]^{-1/2}$, the conditions $\alpha + \beta < 1$ in (8) and (9) ask entirely too little. They can be replaced by $\alpha + \beta \to 0$ as $\epsilon \to 0$. In such cases the conjecture is saved and $\delta + \gamma < 2$ is all that is needed. The perturbation D is added entirely without cost.

Theorem 5.2. *Let (2) – (4), (7)–(9) without $\alpha+\beta < 1$, and (14) hold for (13). For α and β defined in (8) and (9) and for γ and δ defined in (14) let*

$$\gamma + \delta + \alpha + \beta < 2.$$

Then $a \in L^2[0,\infty)$ implies $g(t,x(t)) \in L^2[0,\infty)$.

Proof. Starting with V' in Theorem 5.1 we can find an ξ as small as we please and a correspondingly large M with

$$\begin{aligned}
&V'(t,\epsilon)\\
&\leq 2a(t)g(t,x(t)) - 2g^2(t,x(t)) + \int_0^t |D(t,s)|[g^2(s,x(s)) + g^2(t,x(t))]ds\\
&+ C(t,t-\epsilon)\int_{t-\epsilon}^t [g^2(u,x(u)) + g^2(t,x(t))]du\\
&+ \int_{t-\epsilon}^t C(t,s)[g^2(t,x(t)) + g^2(s,x(s))]ds + \epsilon C_s(t,t-\epsilon)\int_{t-\epsilon}^t g^2(u,x(u))du\\
&+ \delta g^2(t,x(t)) - \int_0^t |D(t,s)|g^2(s,x(s))ds\\
&\leq \xi g^2(t,x(t)) + Ma^2(t) - 2g^2(t,x(t)) + \delta g^2(t,x(t))\\
&+ g^2(t,x(t))\left[\int_0^t |D(t,s)|ds + C(t,t-\epsilon)\int_{t-\epsilon}^t ds + \int_{t-\epsilon}^t C(t,s)ds\right]\\
&+ \int_{t-\epsilon}^t [C(t,t-\epsilon) + C(t,s) + \epsilon C_s(t,t-\epsilon)]g^2(s,x(s))ds\\
&\leq g^2(t,x(t))[\xi - 2 + \delta + \gamma + \epsilon C(t,t-\epsilon) + \int_{t-\epsilon}^t C(t,s)ds] + Ma^2(t)\\
&+ \int_{t-\epsilon}^t [C(t,t-\epsilon) + C(t,s) + \epsilon C_s(t,t-\epsilon)]g^2(s,x(s))ds.
\end{aligned}$$

We then have

$$\begin{aligned}
V'(t,\epsilon) \leq{}& Ma^2(t) + g^2(t,x(t)[\xi - 2 + \delta + \gamma + \beta]\\
&+ \int_{t-\epsilon}^t [C(t,t-\epsilon) + C(t,s) + \epsilon C_s(t,t-\epsilon)]g^2(s,x(s))ds.
\end{aligned}$$

When we integrate from ϵ to t as we did after (11) the last term is bounded by $\alpha\int_0^t g^2(s,x(s))ds$. That integration will now yield the result. $\square$

6. Infinite delay

Volterra recognized two kinds of singularities: a discontinuous kernel or an infinite delay. Here, we consider the combination in the form of

$$x(t) = a(t) - \int_{-\infty}^{t} C(t,s)g(s,x(s))ds \tag{16}$$

where C satisfies (2), (3), and several integrability and limit conditions. Such equations can have three kinds of solutions. Given a continuous initial function $\phi : (-\infty, t_0] \to \Re$ we seek a solution $x(t)$ for $t > t_0$ with $x(t) = \phi(t)$ for $t \le t_0$. If

$$\phi(t_0) = a(t_0) - \int_{-\infty}^{t_0} C(t_0,s)g(s,\phi(s))ds \tag{17}$$

then $x(t)$ is continuous on $(-\infty, t)$ for $t > t_0$; otherwise, the solution has a discontinuity at t_0. It can be shown that there is an ψ arbitrarily near ϕ with $x(t)$ continuous. This gives two kinds of solutions. The third kind occurs when x satisfies (16) on $(-\infty, \infty)$ so at any t_0, then $x(t)$ is its own initial function. Periodic solutions are central examples of this and are discussed throughout [5]. Several existence results using fixed point theory for this kind of equation are found in Burton [3], Section 1 of Chapter 3 of [5], and in Burton and Makay [8].

Much of the work done on this equation using Liapunov functions and a nonsingular kernel is summarized in [5] and interesting recent work is found in Zhang [14].

Here, we again allow for C to be singular and (8) and (9) hold. Moreover, we suppose that ϕ is chosen so that (17) holds and that there is a continuous solution on $[t_0, \infty)$. The work of Burton-Makay [8] speaks extensively to that case when the kernel is nonsingular. Our Liapunov functional will have the form

$$V(t,\epsilon) = \int_{-\infty}^{t-\epsilon} C_s(t,s)\left(\int_s^t g(v,x(v))dv\right)^2 ds. \tag{18}$$

The critical condition which makes this a viable Liapunov functional is

$$\lim_{s\to-\infty} (t-s)C(t,s) = 0 \tag{19}$$

for fixed t. It is used in the integration by parts formula for the derivative and it allows us to skip one term in the usual Liapunov functional.

Theorem 6.1. *Let (2), (3), and (19) hold. If $x(t)$ solves (16) with a bounded and continuous initial function ϕ satisfying (17), then for $\epsilon > 0$ the derivative of V defined by (18) satisfies (6). If (8) and (9) hold and if $|g(t,x)| \le |x|$ then $a \in L^2[0,\infty)$ implies that $g(t,x(t)) \in L^2[t_0,\infty)$.*

Proof. As $C_{st} \le 0$ we have

$$V'(t,\epsilon) \le C_s(t,t-\epsilon)\bigg(\int_{t-\epsilon}^{t} g(v,x(v))dv\bigg)^2 + 2g(t,x(t))\int_{-\infty}^{t-\epsilon} C_s(t,s)\int_s^t g(v,x(v))dvds.$$

If we integrate the last term by parts as we have done before and use (19) on the lower limit with the bounded initial function then we obtain

$$V'(t,\epsilon) \le -2g(t,x(t))\int_{t-\epsilon}^{t} C(t,s)g(s,x(s))ds + C_s(t,t-\epsilon)\bigg(\int_{t-\epsilon}^{t} g(v,x(v))dv\bigg)^2 + 2g(t,x(t))C(t,t-\epsilon)\int_{t-\epsilon}^{t} g(v,x(v))dv + 2g(t,x(t))[a(t)-x(t)].$$

The rest of the proof proceeds just as we have seen several times before. □

We can add the perturbation D and add the Liapunov functional

$$\int_{-\infty}^{t}\int_{t-s}^{\infty} |D(u+s,s)|dug^2(s,x(s))ds$$

and we can find a lower bound on the Liapunov functional as we did in Theorem 3.3. In summary, (1) can be perturbed with a singularity at $t=s$, it can be perturbed with $D(t,s)$, and it can be perturbed to $(-\infty,0]$ without substantially changing the solution. Equation (1) is very stable and it is a defensible practice to use a kernel satisfying (2) to model real-world problems, just as noted by Volterra in 1928.

7. The truncated equation

In this section we consider the equation

$$x(t) = a(t) - \int_{t-h}^{t} C(t,s)g(s,x(s))ds,\ h>0, \tag{20}$$

and we must modify (2) in a crucial way, asking that

$$C(t,s)\ge 0,\ C_{st}(t,s)\le 0,\ C_s(t,s)\ge 0,\ C(t,t-h)=0 \tag{21}$$

when

$$0<\epsilon<h \text{ and } t-h\le s\le t-\epsilon. \tag{22}$$

Theorem 7.1. *Let (21) and (22) hold for (20) and let V be defined by*

$$V(t,\epsilon) = \int_{t-h}^{t-\epsilon} C_s(t,s)\bigg(\int_s^t g(v,x(v))dv\bigg)^2 ds.$$

Then along a continuous solution of (20) $V'(t,\epsilon)$ satisfies (6).

The proof of the theorem and the consequences should now be routine.

Acknowledgment

Thanks go to Profs. Leigh Becker, Tetsuo Furumochi, Geza Makay, John Purnaras, and Bo Zhang for reading, correcting, and commenting on the manuscript.

8. References

1. Burton, T. A., Examples of Lyapunov functionals for non-differentiated equations. *Proc. First World Congress of Nonlinear Analysts, 1992. V. Lakshmikantham, ed. Walter de Gruyter, New York* (1996) 1203-1214.

2. Burton, T. A., Boundedness and periodicity in integral and integro-differential equations *Diff. Eq. Dynamical Systems* **1** (1993) 161–172.

3. Burton, T. A., Differential inequalities and existence theory for differential, integral, and delay equations, pp. 35-56. In "Comparison Methods and Stability Theory." Xinzhi Liu and David Siegel, eds, Dekker, New York, 1994.

4. Burton, T. A., Scalar nonlinear integral equations *Tatra Mt. Math. Publ.* **38** (2007) 41–56.

5. Burton, T. A., *Liapunov Functionals for Integral Equations* Trafford, Victoria, B. C., Canada, 2008. (www.trafford.com/08-1365)

6. Burton, T. A., A Liapunov functional for a linear integral equation *Electronic J. Qualitative Theory Diff. Equations* **10** (2010), 1-10. (http://www.math.u-szeged.hu/ejqtde)

7. Burton, T. A. and Dwiggins, D. P., Resolvents, integral equations, and limit sets, *Mathematica Bohemica*, to appear.

8. Burton, T. A. and Makay, G., Continuity, compactness, fixed points, and integral equations. Electronic J. Qualitative Theory of Diff. Equ. **14** (2002), 1-13.(http://www.math.u-szeged.hu/ejqtde)

9. Kirk, C. M. and Olmstead, W. E., Blow-up solutions of the two-dimensional heat equation due to a localized moving source, Analysis and Applications, 3(2005), 1-16.

10. Levin, J. J., The asymptotic behavior of the solution of a Volterra equation *Proc. Amer. Math. Soc.* **14** (1963) 534–541.

11. Miller, Richard K., *Nonlinear Volterra Integral Equations* Benjamin, Menlo Park, CA, 1971.

12. Volterra, V., Sur la théorie mathématique des phénomès héréditaires. *J. Math. Pur. Appl.***7** (1928) 249–298.

13. Windsor, A., A contraction mapping proof of the smooth dependence on parameters of solutions to Volterra integral equations, Nonlinear Analysis: Theory, Methods and Applications, **72**(2010), 3627-3634.

14. Zhang, Bo, Boundedness and global attractivity of solutions for a system of nonlinear integral equations. *Cubo: A Mathematical Journal* **11** (2009) 41–53.

Northwest Research Institute, 732 Caroline St., Port Angeles, Washington, USA.
E-mail address: taburton@olypen.com