Section 20: Arrow Diagrams on the Integers

Most of the material we have discussed so far concerns the idea and representations of functions. A function is a relationship between a set of inputs (the “leave from” buildings in our cities) we called the domain, and a set of outputs (the “go to” buildings) called the range. One of the representations we used for a function was an arrow diagram. In this section, we will extend our ideas to arrow diagrams on the set of integers. Before we do this, let’s recall what the property of an arrow diagram of a function is:

An arrow diagram is the arrow diagram for a function if every number in the domain has one and only one arrow leaving from it.

There is nothing in this property that says anything about the size of the domain or the range. As long as this property is satisfied, then we have the arrow diagram of a function.

The set of integers is the set \{..., -4, -3, -2, -1, 0, 1, 2, 3, 4,...\}. In mathematics we use “...” to represent the same idea as when we say in English, “and so on”. In other words, there is a pattern to something, and the “...” means to just continue doing the pattern. In the set above, the pattern on the left is, “keep subtracting 1”, and the pattern on the right is, “keep adding 1”.

The Road Coloring Problem

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How big is the set of integers? In mathematics, we say the size of a set is finite (finite) if we can count the number of elements in it. But the set of integers is too big to count. There is no integer large enough for its size. If we tried to pick a number for its size, there would always be more numbers in the set of integers than the number we picked. For this reason, we say the set of integers is not finite. The way mathematicians say “not finite” is infinite (infinite).

But just because the set of integers is infinite, doesn’t mean that we can’t construct arrow diagrams on it. Here is what we’ll do. Let’s list two sets of integers in column form, just like we did with the arrow diagrams for the road coloring problem. We’ll use the “...” on the top and bottom to mean “and so on”. The column on the left will represent our domain (the “leave from” numbers) and the column on the right will contain our range (the “go to” numbers). Then we will connect the two columns with arrows, just as we did before.

Here is one arrow diagram on the integers...
Since we can’t show all of the integers, we have to imagine the pattern that exists for the part that we can’t show. We have to imagine “all” of the arrows being there, and try to picture the arrows that connect the numbers that are not listed. Can you determine the “go to” number for the arrow that starts at 10 (that is, has 10 as its domain element or “leave from” number)?

How about the one that starts at 25? Or 100? Or 1000? Can you also do the negative numbers? How about -10? Where does its arrow lead to? What about -100?
Do you see that there is always the same pattern, no matter where we start in the domain? We can take advantage of this and find a way to express the relationship between the domain and range.

We will use the same notation as we did for the road coloring functions. Let's use the variable “x” to indicate the input number (the 'leave from' number) and the variable “y” for the output number (the “go to” number). Now we will try to write an equation that relates each input to its output. If we write some inputs and outputs in column form, this may help.

<table>
<thead>
<tr>
<th>Input</th>
<th>Output</th>
</tr>
</thead>
<tbody>
<tr>
<td>x</td>
<td>y</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>10</td>
<td>11</td>
</tr>
<tr>
<td>100</td>
<td>101</td>
</tr>
<tr>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>-10</td>
<td>-9</td>
</tr>
</tbody>
</table>
Just as we did with the road functions, we can also list our input and outputs as ordered pairs in the form \((x, y)\). Remember, our input or \(x\) variable is the first coordinate and our output or \(y\) variable is the second coordinate.

\[\{(-1, 0), (0, 1), (1, 2), (2, 3), (3, 4), \ldots\}\]

Remember how ordered pairs become points plotted on a Cartesian coordinate plane? Let’s construct that representation also.

Have you seen the pattern yet? Every output is one more than its input. When the relationship between the domain and range is this simple, we have another representation we can use called an equation. For this function, the equation that represents the relationship between the input \(x\), and the output \(y\) is:

\[y = x + 1\]
Just as we did with the road functions, we can also list our input and outputs as ordered pairs in the form \((x, y)\). Remember, our input or \(x\) variable is the first coordinate and our output or \(y\) variable is the second coordinate.

Here is another arrow diagram on the integers. Can you determine the pattern between the domain and the range? Can you represent this arrow diagram as ordered pairs, and as points in a Cartesian plane?
Challenge

Exercise 20.1

As a team, using the arrow diagram on the previous page, write some inputs and outputs in column form, express these inputs and outputs as a set of ordered pairs, and plot the ordered pairs on a Cartesian coordinate plane. Finally, find an equation to represent the relationship between the input variable \( x \), and the output variable \( y \). Each team should report their work to the class.

<table>
<thead>
<tr>
<th>Input</th>
<th>Output</th>
</tr>
</thead>
<tbody>
<tr>
<td>x</td>
<td>y</td>
</tr>
</tbody>
</table>

Put your ordered pairs for the function in this box.
Plot your ordered pairs below. Make sure you label the axes appropriately.

Can you find the equation which relates the input $x$, to the output $y$? If so, write it below.
The arrow diagrams we have been working with are, in many ways, the exceptions. Most functions on the integers will not have an equation that relates the input variable $x$ with the output variable $y$. For example, here is another arrow diagram on the integers:

\[
\begin{array}{c}
4 \quad 4 \\
3 \quad 3 \\
2 \quad 2 \\
1 \quad 1 \\
0 \quad 0 \\
-1 \quad -1 \\
-2 \quad -2 \\
-3 \quad -3 \\
-4 \quad -4 \\
\end{array}
\]

Do you see the pattern in this arrow diagram? Don’t worry if you can’t. No one can. There is no pattern to the relationship between inputs and outputs for this arrow diagram. Therefore we can not find an equation for this function.

Luckily, the most important functions that are used to solve problems in science and engineering do have patterns that relate their inputs and outputs, and so we can find equations for them. In many ways, science is all about finding patterns. If you haven’t tried this before, it may take some time to do it well. But with practice, you can become good at it. All human beings have amazing pattern recognition capabilities. We would not be able to understand the things we see or hear if we could not recognize patterns.
The arrow diagrams we have been working with are, in many ways, the exceptions. Most functions on the integers will not have an equation that relates the input variable $x$ with the output variable $y$. For example, here is another arrow diagram on the integers:

Do you see the pattern in this arrow diagram? Don’t worry if you can’t. No one can. There is no pattern to the relationship between inputs and outputs for this arrow diagram. Therefore we can not find an equation for this function.

Just as we did with the road functions, we will need to be able to understand and produce all of the representations for a function from whichever representation we are given. What we want to do is start with a given equation, and produce an arrow diagram, ordered pairs, and a graph.

Let’s do an example. Suppose we are given the equation $y = x - 1$.

The best way to start constructing the representations is to think of the equation as an “input-output machine”. In other words, if you give the machine a certain input, what will its output be? We will set up two columns: one for inputs, $x$, and one for outputs, $y$. Then we substitute each $x$ value in the equation and determine the $y$ value that corresponds to it. For example, if we substitute 4 for $x$ in the equation $y = x - 1$, we get $y = 4 - 1$. Thus $y = 3$. This means that for the input 4, we get the output 3. Below we list some more inputs and output for the equation.

<table>
<thead>
<tr>
<th>Input ($x$)</th>
<th>Output ($y$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>3</td>
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<tr>
<td>3</td>
<td>2</td>
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<tr>
<td>2</td>
<td>1</td>
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<td>1</td>
<td>0</td>
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<tr>
<td>0</td>
<td>-1</td>
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<td>-1</td>
<td>-2</td>
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<td>-3</td>
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<tr>
<td>-3</td>
<td>-4</td>
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<tr>
<td>-4</td>
<td>-5</td>
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</tbody>
</table>
These inputs and outputs can now become a set of ordered pairs.

\{(4,3), (3,2), (2,1), (1, 0), (0,1), (-1, -2), (-2,-3), (-3, -4), (-4, -5)\}

Now we can take these ordered pairs and construct the arrow diagram and the graph of this function.
Exercise 20.2

For each of the equations below, write down columns of inputs and outputs using the equation and convert the input-output columns into ordered pairs. Next represent the function as both an arrow diagram and a graph of points. Report your results to the class.

\[ y = x + 2 \]

<table>
<thead>
<tr>
<th>Input</th>
<th>Output</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x )</td>
<td>( y )</td>
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</table>

Put your ordered pair in the box below.
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<table>
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<tbody>
<tr>
<td>4</td>
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<tr>
<td>-4</td>
<td>-4</td>
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</table>

The Road Coloring Problem
Exercise 20.3

For each of the equations below, write down columns of inputs and outputs using the equation and convert the input-output columns into ordered pairs. Next represent the function as both an arrow diagram and a graph of points. Report your results to the class.

\[ y = x^2 \] (remember: \( x^2 \) means \( x \cdot x \))

<table>
<thead>
<tr>
<th>Input ( x )</th>
<th>Output ( y )</th>
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</thead>
<tbody>
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<td></td>
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</table>

Put your ordered pair in the box below.