In our last section, we defined a new operation for the arrow diagrams. In this section we will take advantage of the relationship between the arrow diagrams and the road matrices to find an equivalent operation for the matrices.

What really happens when we compose the arrow diagrams together? Suppose we were given the following two arrow diagrams. To save space we will stop writing “leave from” and “go to” above their columns in the arrow diagrams. By now you should know that the arrow always leaves from the “leave from” building and goes to the “go to” building.

It really doesn’t matter if we use the arrow diagrams or the road matrices to represent what is happening. We could just as easily write the matrices for the arrow diagrams. Whenever we write the matrix for a road arrow diagram we will put brackets around the name so that you know that it is a matrix. There is a special correspondence between the road arrow diagrams and the road matrices that we will study.

The figures below show this correspondence between these two representations.
The composition of the blue and red arrow diagrams is below, and to its right is the product of the two matrices $[B]$ and $[R]$. We will define $[B] \ast [R]$ to be the same matrix as $[B \circ R]$.

Since we will be combining matrices together we will need a symbol for this operation. We will use “$\ast$”. This new operation for matrices is equivalent to composition of the arrow diagrams. We will refer to this operation as multiplication of the matrices.
Another example might be helpful.
The easiest way to multiply road matrices is to convert them back into arrow diagrams, compose the arrow diagrams, and then convert the combined arrow diagram back into a matrix. In other words, if we start with these matrices.

\[
[R] = \begin{bmatrix}
1 & 2 & 3 \\
1 & 0 & 1 & 0 \\
2 & 0 & 0 & 1 \\
3 & 1 & 0 & 0 \\
\end{bmatrix}
\]

\[
[B] = \begin{bmatrix}
1 & 2 & 3 \\
1 & 0 & 1 & 0 \\
2 & 1 & 0 & 0 \\
3 & 1 & 0 & 0 \\
\end{bmatrix}
\]

Then to find \([R] \ast [B]\), we convert them into arrow diagrams and compose the arrow diagrams together.

\[
[R] \ast [B] = \begin{bmatrix}
1 & 2 & 3 \\
1 & 1 & 0 & 0 \\
2 & 1 & 0 & 0 \\
3 & 0 & 1 & 0 \\
\end{bmatrix}
\]
Composing the arrow diagrams together is easy, since all you have to do is follow the arrows. Matrix multiplication is a little trickier, but let’s see if we can begin to understand it better. To do this, we will concentrate on the movement of one person in our city. Let’s look at Candice who starts in building 1, and follows the instruction “red-blue”.

Candice leaves from building 1 and goes to building 2 using the red road. Therefore, we have a 1 in the 1st row, 2nd column position of $[R]$. Next Candice leaves from building 2 and goes to building 1 using the blue road, so there is a 1 in the 2nd row, 1st column position of $[B]$.
Looking at the matrices we see that the **row** of the **first matrix** tells us where we originally started and the **column** of the **second matrix** tells us where we finish. The flow chart below tells the story of Candice’s movements after following the red and blue commands.

The “go to” building becomes the “leave from” building.

This “1” means Candice traveled from building 1 back to building 1 by first following R and then following B.
Now let’s look at what happens to Miguel when the command “Red-Blue” is called.

Miguel starts in building 2. He leaves building 2 and goes to building 3. Miguel then leaves from building 3 and goes to building 1. If we wanted to use the road matrices to tell the same story, it would look like this.

Miguel "goes to" building 3.
Miguel "leaves from" building 3 and follows the blue road.
Miguel "goes to" building 1.

Miguel “goes to” building 3.
Miguel “leaves from” building 3 and follows the blue road.
Miguel “goes to” building 1.

Now let’s look at what happens to Miguel when the command “Red-Blue” is called.

Miguel starts in building 2. He leaves building 2 and goes to building 3. Miguel then leaves from building 3 and goes to building 1. If we wanted to use the road matrices to tell the same story, it would look like this.

Miguel "goes to" building 3.
Miguel "leaves from" building 3 and follows the blue road.
Miguel "goes to" building 1.

The “go to” building becomes the “leave from” building.

This "1" means Miguel traveled from building 2 to building 1 by first following R and then following B.
Exercise 17.1:
Use the road matrices to tell the story of what happens to Marcus when the “Red-Blue” command is called. Follow the same format that we used for Candice and Miguel. Fill in the spaces provided with an explanation of how Marcus moved. Make sure you put the arrows in their correct places.
The Road Coloring Problem

\[
\begin{bmatrix}
1 & 0 & 1 & 0 \\
2 & 0 & 0 & 1 \\
3 & 1 & 0 & 0 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 1 & 0 & 0 \\
2 & 1 & 0 & 0 \\
3 & 0 & 1 & 0 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
1 & 0 & 0 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
1 & 0 & 0 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 2 & 3 \\
1 & 0 & 1 \\
2 & 0 & 0 \\
3 & 1 & 0 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 0 & 1 \\
2 & 0 & 0 \\
3 & 1 & 0 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
1 & 0 & 0 \\
\end{bmatrix}
\]
Exercise 17.2: Matrix Multiplication

Directions: For each city below, draw the arrow diagram for the road functions, find the associated matrices, then multiply the given matrices together by combining the associated arrow diagrams.

__________________ Teacher __________________ Date _____

\[
\begin{align*}
B & = \\
\begin{bmatrix}
1 & 1 \\
2 & 2 \\
3 & 3 \\
\end{bmatrix} & \quad \begin{bmatrix}
1 & 1 \\
2 & 2 \\
3 & 3 \\
\end{bmatrix} & \quad \begin{bmatrix}
1 & 1 \\
2 & 2 \\
3 & 3 \\
\end{bmatrix} \\
R & = \\
\end{align*}
\]

\[
B \times R = \\
\begin{bmatrix}
1 & 1 \\
2 & 2 \\
3 & 3 \\
\end{bmatrix} \times \\
\begin{bmatrix}
1 & 1 \\
2 & 2 \\
3 & 3 \\
\end{bmatrix} = \\
\begin{bmatrix}
\_ & \_ & \_ \\
\_ & \_ & \_ \\
\_ & \_ & \_ \\
\end{bmatrix}
\]
Exercise 17.3  Matrix Multiplication

Directions: For each city below, draw the arrow diagram for the road functions, find the associated matrices, then multiply the given matrices together by combining the associated arrow diagrams.

__________________ Teacher __________________ Date _____

\[
\begin{bmatrix}
1 & 1 \\
2 & 2 \\
3 & 3 \\
4 & 4 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 1 \\
2 & 2 \\
3 & 3 \\
4 & 4 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 1 \\
2 & 2 \\
3 & 3 \\
4 & 4 \\
\end{bmatrix}
\] \[
\begin{bmatrix}
1 & 1 \\
2 & 2 \\
3 & 3 \\
4 & 4 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 1 \\
2 & 2 \\
3 & 3 \\
4 & 4 \\
\end{bmatrix}
\] \[
\begin{bmatrix}
1 & 1 \\
2 & 2 \\
3 & 3 \\
4 & 4 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 1 \\
2 & 2 \\
3 & 3 \\
4 & 4 \\
\end{bmatrix}
\] \[
\begin{bmatrix}
1 & 1 \\
2 & 2 \\
3 & 3 \\
4 & 4 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 1 \\
2 & 2 \\
3 & 3 \\
4 & 4 \\
\end{bmatrix}
\] \[
\begin{bmatrix}
1 & 1 \\
2 & 2 \\
3 & 3 \\
4 & 4 \\
\end{bmatrix}
\]
**Exercise 17.4  Function composition**

Directions: Complete the table below by composing the arrow diagrams of the functions.

<table>
<thead>
<tr>
<th>Composition</th>
<th>1 → 1</th>
<th>2 → 2</th>
<th>3 → 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 → 1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2 → 2</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3 → 3</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Exercise 17.5  Matrix multiplication

Directions: Complete the table below by multiplying the matrices. Use the table from exercise 17.4.

<table>
<thead>
<tr>
<th>Name __________________</th>
<th>Teacher __________________</th>
<th>Date _____</th>
</tr>
</thead>
</table>

<table>
<thead>
<tr>
<th>Matrix Multiplication</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>*</td>
<td>1 0 0</td>
</tr>
<tr>
<td></td>
<td>0 1 0</td>
</tr>
<tr>
<td></td>
<td>0 0 1</td>
</tr>
<tr>
<td>1 0 0</td>
<td>0 1 0</td>
</tr>
<tr>
<td>0 1 0</td>
<td>0 0 1</td>
</tr>
<tr>
<td>0 0 1</td>
<td>1 0 0</td>
</tr>
<tr>
<td>1 1 0</td>
<td>0 1 0</td>
</tr>
<tr>
<td>1 0 1</td>
<td>0 0 1</td>
</tr>
<tr>
<td>0 0 1</td>
<td>1 0 0</td>
</tr>
<tr>
<td>0 0 1</td>
<td>1 0 0</td>
</tr>
</tbody>
</table>

The Road Coloring Problem

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